ME 526 - Homework 2

Mathematica analysis of heated spherical egg

Due Date:

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(Group № 2)

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Introduction:

This homework exercise will look at modeling steady state conduction heat transfer through solids. We will look at 3 heat transfer situations: 1-D heat conduction in a cuboid, complex heat conduction in the cuboid, and heat conduction in the sphere. For first and last cases the analytic solutions will be derived followed by COMSOL application simulations. For the second case the derivation of analytical solution is a complex problem thus we will only complete the COMSOL simulation.

Problem #1: (3-D, Steady state, Cuboid)

A cuboid, with one corner at the origin and lengths: L_x, L_y, L_z. The surface at $z = L_z$ has temperature T_2 and the surface at $z = 0$ has temperature T_1. The other surfaces are perfectly insulating. Steady State. The material has K, ρ, c all constants.

1) First, derive an analytical answer for T:

(Give all the arguments. Use talk on the cylinder as a basis. This is certainly an easier problem than for the cylinder one, but I still want the arguments. What happens to κ ? Why?)

$$q = -K\nabla T \quad ; \quad (1)$$

$$\frac{\delta}{\delta x}\left(K\frac{\delta T}{\delta x}\right) + \frac{\delta}{\delta y}\left(K\frac{\delta T}{\delta y}\right) + \frac{\delta}{\delta z}\left(K\frac{\delta T}{\delta z}\right) + q^{\cdot} = \rho c_p \frac{\delta T}{\delta t}$$

; (2)

$$\kappa = \frac{K}{\rho c_p} \quad ; \quad (3)$$

thermal conductivity is constant: $K = C$

No heat generation: $q^{\cdot} = 0$

$$\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2} = \frac{\rho c_p}{K} \frac{\delta T}{\delta t} \quad ; \quad (2)$$

Steady state reached $\frac{\delta T}{\delta t} = 0$.

$$\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2} = 0 \quad ; \quad (2)$$

Insulation condition around all 4 surfaces in y and x planes implies:

$$q_x = 0 \; ; \; q_y = 0 \; ; \; \frac{\delta T}{\delta x} = 0 \; ; \; \frac{\delta T}{\delta y} = 0 \; ; \; \frac{\delta^2 T}{\delta x^2} = 0 \; ; \; \frac{\delta^2 T}{\delta y^2} = 0$$

Thus:

$$\frac{\delta^2 T}{\delta z^2} = 0 \;\; ; \;\; (2)$$

Integrating expression:

$$\frac{\delta T}{\delta z} = C_1$$

Integrating expression:

$$T(z) = C_1 z + C_2$$

Boundary conditions:

$$T(z = 0) = T_1$$

$$T(z = L_z) = T_2$$

Substitute boundary conditions:

$$T_1 = C_1 \times 0 + C_2$$

$$C_2 = T_1$$

$$T_2 = C_1 L_z + T_1$$

$$C_1 = \frac{T_2 - T_1}{L_z}$$

$$T(z) = \frac{T_2 - T_1}{L_z} z + T_1;$$

Substituting values in the expression:

$$T_1 = 200\,K\ ;\ T_2 = 400\,K\ ;\ L_z = 1.0\,m$$

$$T(z) = 200\,z + 200\ ;$$

2) Second, using COMSOL, do the simulation in 3D:

Values for variables required by COMSOL are illustrated below:

$$\rho = 8700\,\frac{kg}{m^3}\ ;\ K = 400\,\frac{W}{m\,K}\ ;\ c = 385\,\frac{J}{kg\,K}\ ;\ (material\ copper)$$

$$T_1 = 200\,K\ \ (z = 0\,m);\ T_2 = 400\,K\ \ (z = L_z)$$

$$L_x = 1.0\,m \;;\; L_y = 0.5\,m \;;\; L_z = 1.0\,m$$

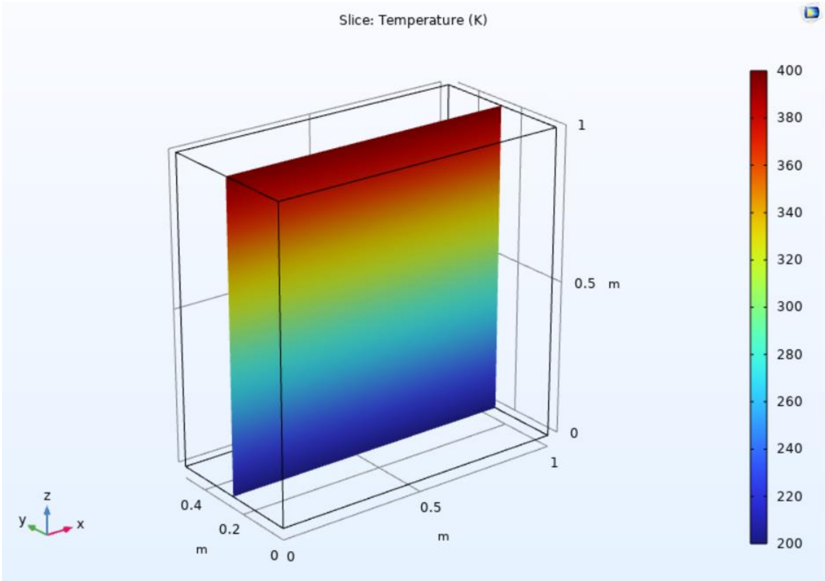


Figure 1.1: (Problem 1) 2-D Slice along the plane at y = 0.25 m. (3-D illustration)

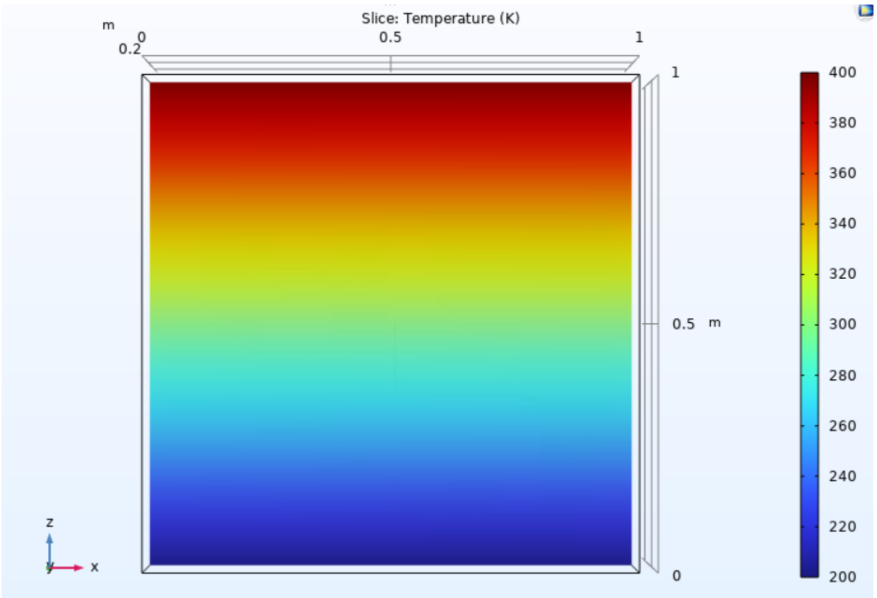


Figure 1.2: (Problem 1) 2-D Slice along the plane at y = 0.25 m.

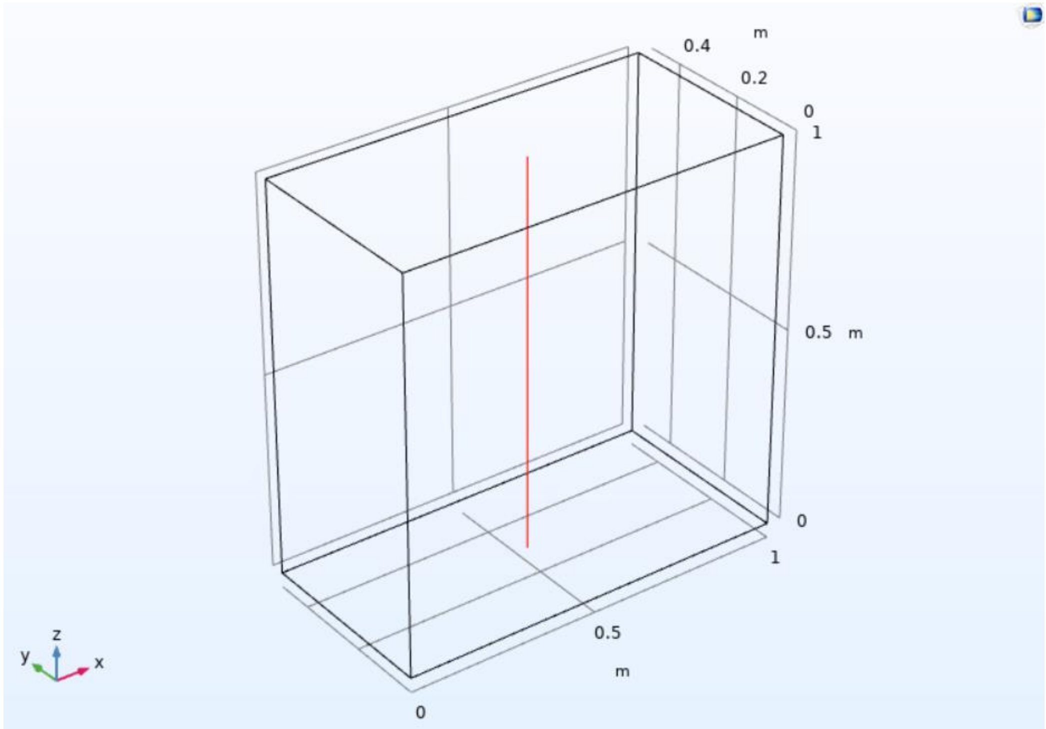


Figure 2.1: 1-D line plot along the center from z = 0 to z = 1. (Trajectory 3-D illustration)

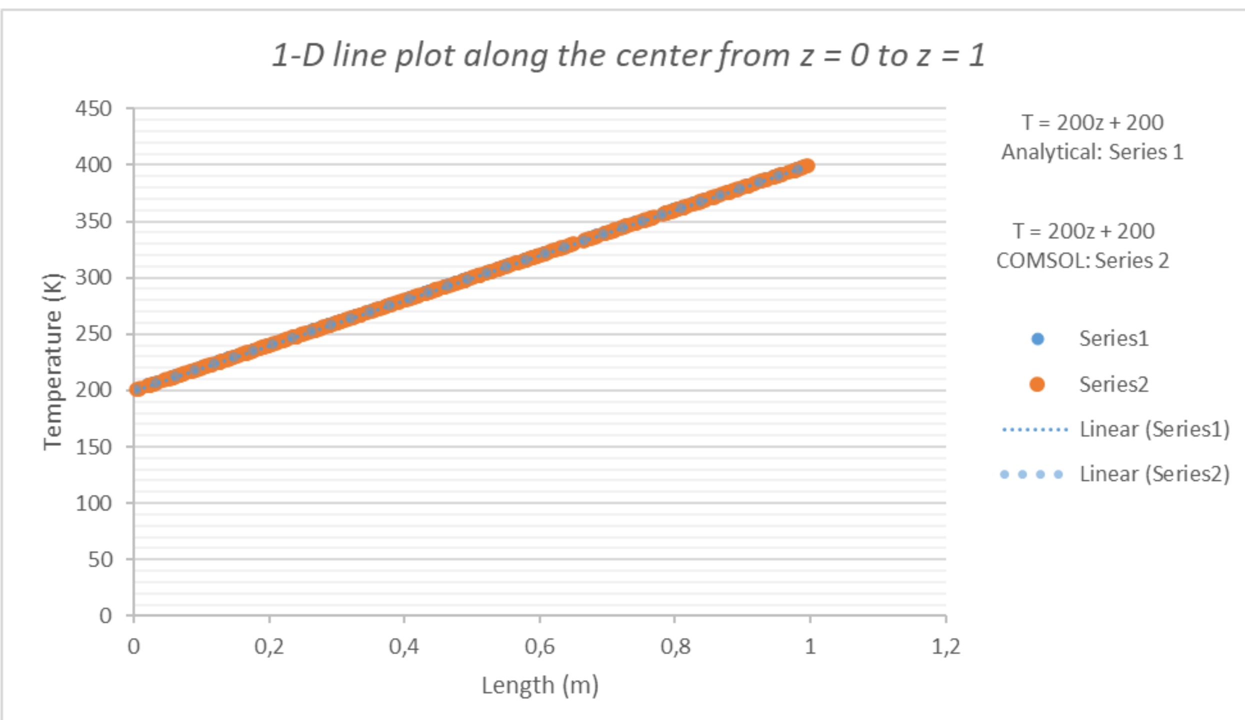


Figure 2.2: 1-D line plot along the center from z = 0 to z = 1.

(Analytical in blue, COMSOL in orange)

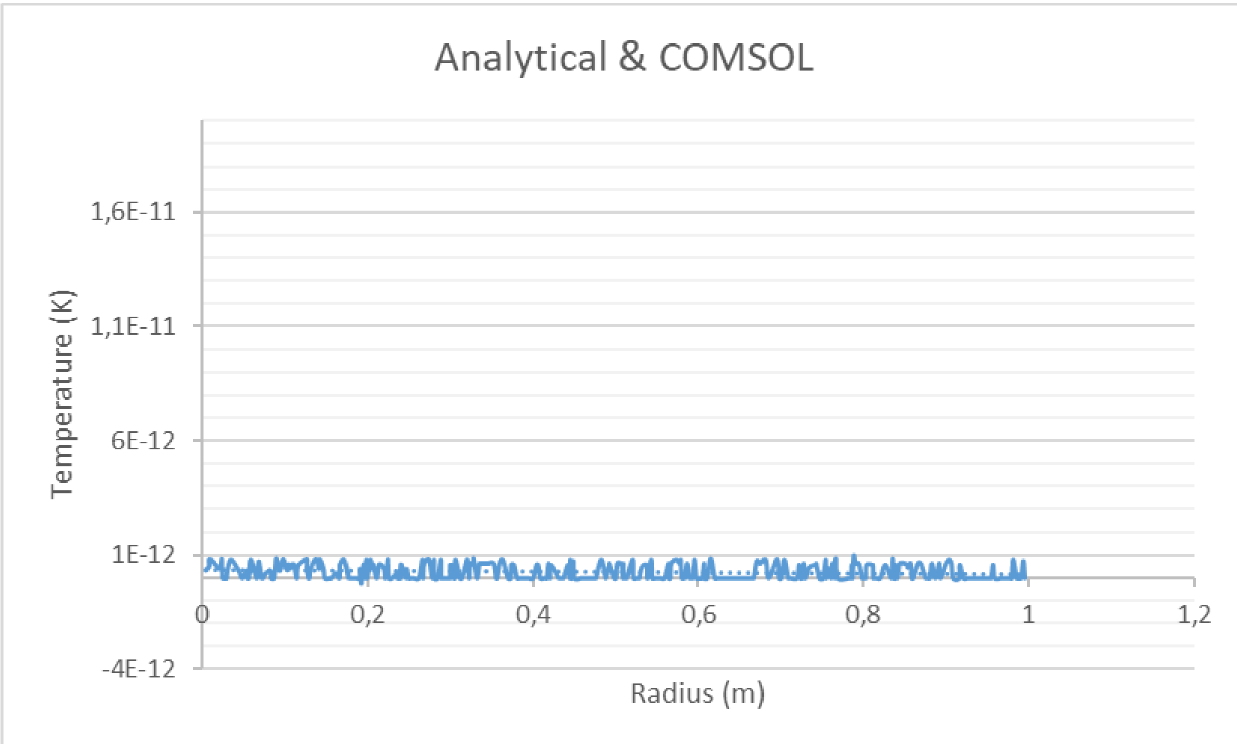


Figure 2.3: Plot of analytical and COMSOL difference in temperature against radius (From Plot 2.2).

Do your plots agree with the analytic result?

Our COMSOL data perfectly agrees with the analytical result. In Figure 2.2, the two plots almost perfectly overlap. Figure 2.3 illustrates the difference between 2 plots which is of a magnitude 10^{-12} (almost negligible).

In this particular case, what are the advantages and disadvantages of the analytic case ?

In this case it is very easy to obtain analytic expression. Simple 1-D conduction heat transfer. The plot's result would be faster obtained analytically, less computational power would be required, less labor and modeling costs.

Disadvantages include knowledge of heat transfer equations. Not everyone is able to derive exact expressions, wrong derivation may result in significant error. In contrast, COMSOL does not need the exact equation to generate data.

Problem #2: (Problem 1 with new initial condition)

Everything same as in problem 1 but additionally, halfway up each side, imagine a line width w and temperature T_3 goes around the box in the x-y plane.

1) Can you solve this analytically? Why or why not?

This problem would be very hard to be solved analytically because it involves heat transfer in 3-D, with very complicated boundary conditions (the surfaces in x and y places are partly at T_3 and partly insulated) Thus this homework decides to skip the derivation part and proceeds immediately into the simulation.

2) Using COMSOL, do the simulation in 3D:

B.C. same as in first problem with additional:

$$w = 0.1\,m \; ; \; T_3 = 500\,K$$

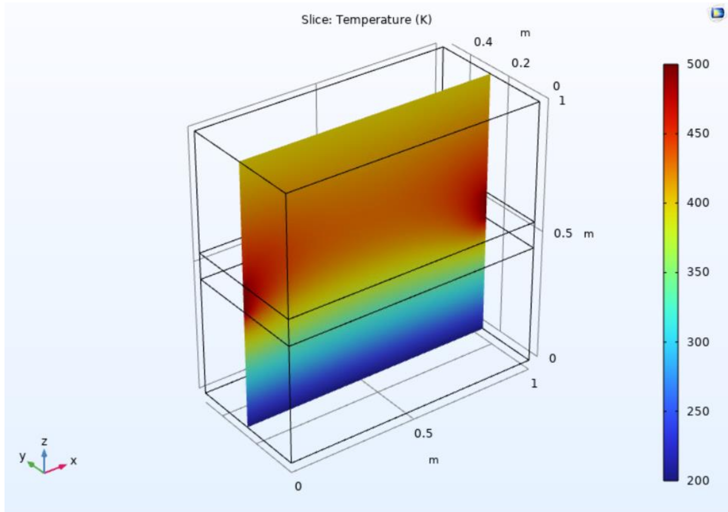


Figure 3.1: (Problem 2) 2-D Slice along the plane at y = 0.25 m (3-D illustration)

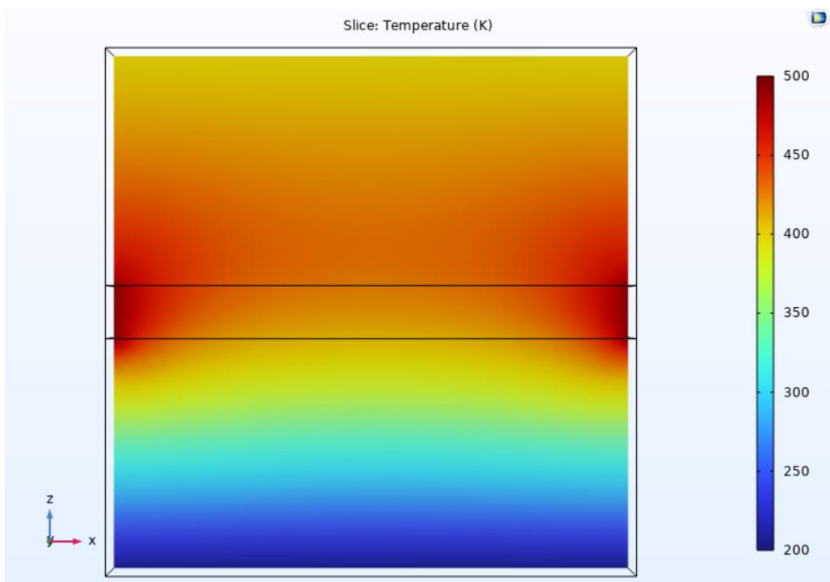


Figure 3.2: (Problem 2) 2-D Slice along the plane at y = 0.25 m.

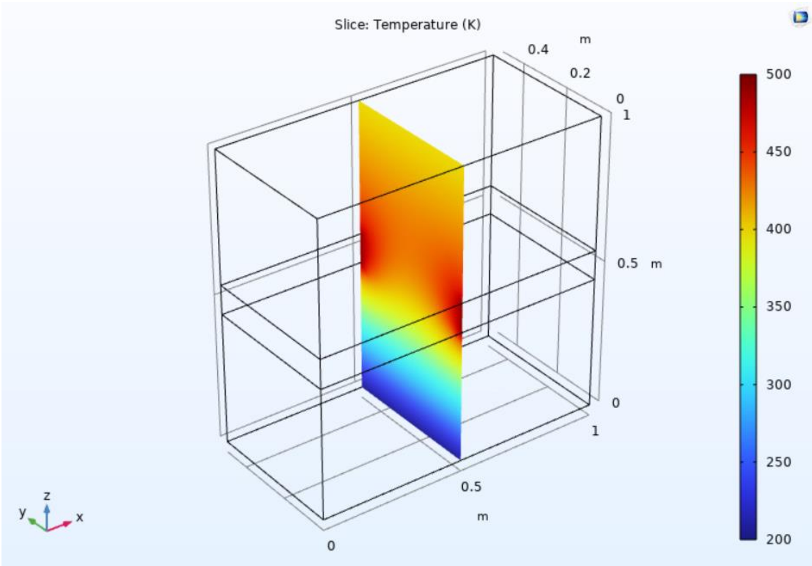


Figure 4.1: (Problem 2) 2-D Slice along the center plane at x = 0.5 m. (3-D illustration)

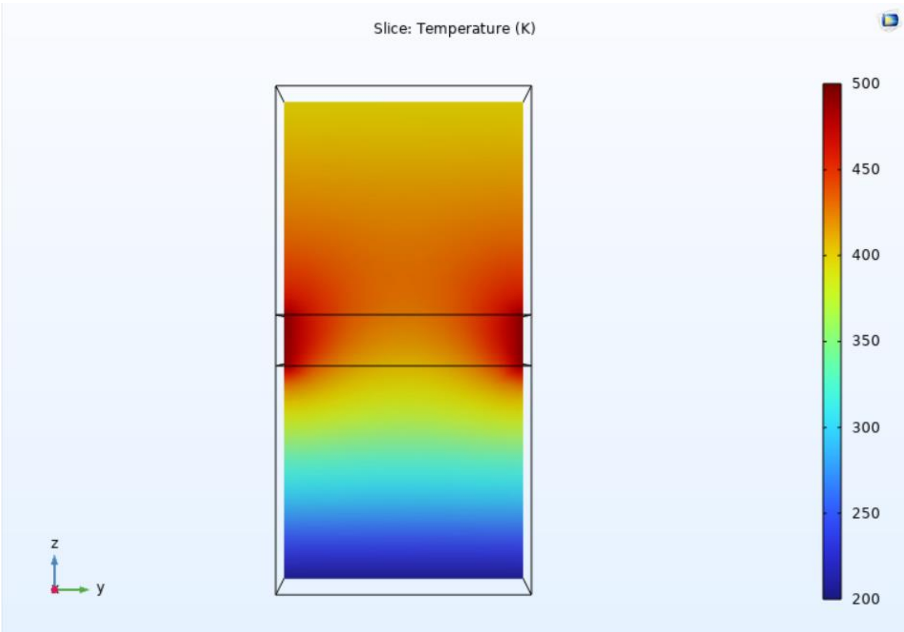


Figure 4.2: (Problem 2) 2-D Slice along the center plane at x = 0.5 m.

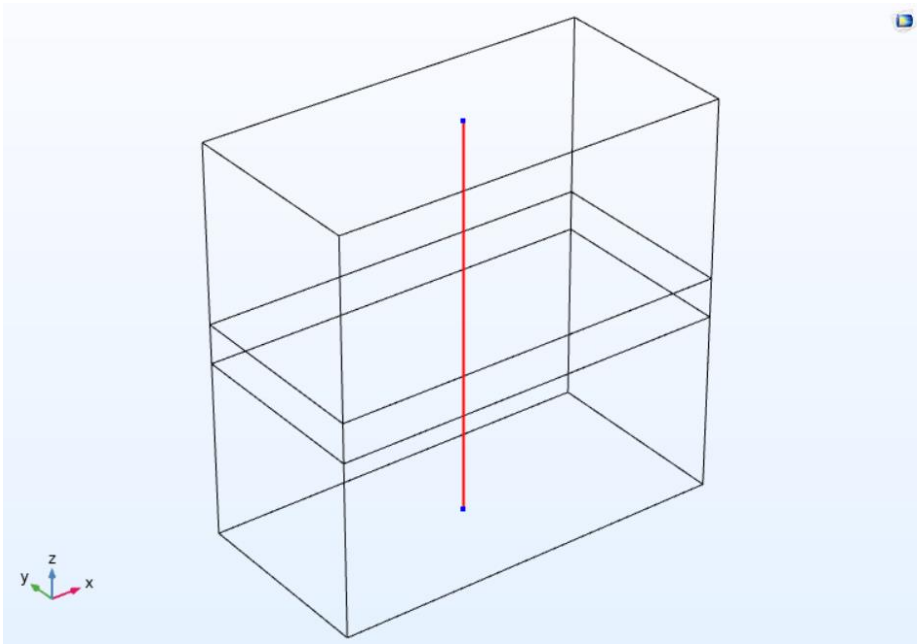


Figure 5.1: (Problem 2) 1-D line plot along the center from z = 0 to z = 1. (Trajectory 3-D illustration)

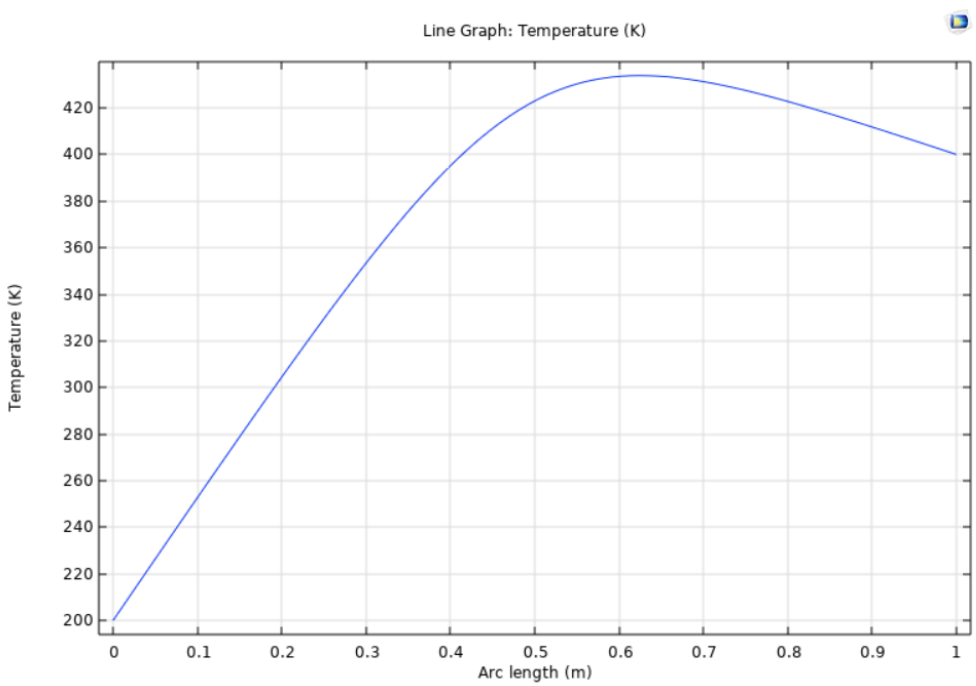


Figure 5.2: (Problem 2) 1-D line plot along the center from z = 0 to z = 1.

Problem #3: (Spheres)

Consider a hollow sphere with: R_{out} , R_{in}. Material has K, ρ, c all constant properties. Boundary conditions are defined in the following way:

T_{in} - inner surface temperature.

$q = -K\frac{dT}{dr}$ - outward heat flux.

1) Solve the problem analytically:

Find $T(r) = f(T_{in}, q, K, R_{out}, R_{in}, r)$

Heat Conduction Equation in Spherical Coordinates:

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left(Kr^2\frac{\delta T}{\delta r}\right)+\frac{1}{r^2 sin^2(\theta)}\frac{\delta}{\delta\phi}\left(K\frac{\delta T}{\delta\phi}\right)+$$

$$\frac{1}{r^2 sin(\theta)}\frac{\delta}{\delta\theta}\left(Ksin(\theta)\frac{\delta T}{\delta\theta}\right)+q^{\cdot}=\rho c_p\frac{\delta T}{\delta t}\ ;\ (5)$$

thermal conductivity is constant: $K = C$

No heat generation: $q^{\cdot} = 0$

Steady state reached $\frac{\delta T}{\delta t} = 0$.

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta T}{\delta r}\right)+\frac{1}{r^2 sin^2(\theta)}\frac{\delta}{\delta\phi}\left(\frac{\delta T}{\delta\phi}\right)+$$

$$\frac{1}{r^2 sin(\theta)}\ \frac{\delta}{\delta\theta}\left(sin(\theta)\frac{\delta T}{\delta\theta}\right)=0 \ ;$$

There is no heat flux in θ , ϕ directions: $q_{\theta} = 0$; $q_{\phi} = 0$ $\frac{\delta T}{\delta\phi} = 0$; $\frac{\delta T}{\delta\theta} =$

$0;$

$$\frac{1}{r^2}\frac{\delta}{\delta r}\left(r^2\frac{\delta T}{\delta r}\right)=0 \ ;$$

Therefore:

$$\frac{\delta}{\delta r}\left(r^2\frac{\delta T}{\delta r}\right)=0\ ;$$

$r^2\frac{\delta T}{\delta r}$ should be a constant (C_1) , $\frac{\delta T}{\delta r}=\frac{C_1}{r^2}$ (6) ;

Boundary Conditions:

1) $q(r = R_{out}) = q_{out} = -K\frac{\delta T(R_{out})}{\delta r}$

$$\frac{\delta T\left(R_{out}\right)}{\delta r} = \frac{C_1}{R_{out}^{\;2}} = -\frac{q_{out}}{K} \;;$$

$$C_1 = -\frac{q_{out} R_{out}^{\;2}}{K} \; (7) \;;$$

2) $T(R_{in}) = T_{in}$

Continuing from (6) , $\int_{T_{in}}^{T(r)} dT = \int_{R_{in}}^{r} \frac{C_1}{r^2} dr$

$$T(r) - T_{in} = -\frac{C_1}{r} + \frac{C_1}{R_{in}}$$

$$T(r) = T_{in} + C_1 \left(\frac{1}{R_{in}} - \frac{1}{r}\right) (8) \;;$$

Sub (7) in (8),

$$T(r) = T_{in} - \frac{q_{out} R_{out}^{\;2}}{K} \left(\frac{1}{R_{in}} - \frac{1}{r}\right)$$

Plug in the values:

$$T_{in} = 500\,K; \;\; q_{out} = 100{,}000 \frac{W}{m^2}; \;\; R_{out} = 0.1\,m; \;\; R_{in} = 0.01\,m; \;\; K = 400 \frac{W}{m\,K}$$

$$T(r) = 500 - 2.5\left(100 - \frac{1}{r}\right)$$

$$T(r) = 250 + \left(\frac{2.5}{r}\right)$$

2) Solve In COMSOL:

$$\rho = 8700\ \frac{kg}{m^3}\ ;\ K = 400\ \frac{W}{m\ K}\ ;\ c = 385\ \frac{J}{kg\ K}\ ;\ (material\ copper)$$

$$T_{in} = 500\ K;\ q_{out} = 100{,}000 \frac{W}{m^2};\ R_{out} = 0.1\ m;\ R_{in} = 0.01\ m;$$

What does an inward or outward heat flux mean? Is q = +100,000 $\frac{W}{m^2}$ making the sphere colder or hotter?

Inward heat flux means that the heat energy is transferred to the body, and the body is getting hotter. Outward heat flux means that the heat energy is transferred from the body, and the body is getting colder. q = +100,000 $\frac{W}{m^2}$ means that the heat flow is leaving the body at this per area rate, therefore the body is getting colder.

How in the world could we arrange this to happen (experimentally I mean, just roughly - give the idea)?

Some kind of chemical reaction that produces constantly the same amount of energy could be taking place inside the sphere. The sphere is put in a constant environment and is hung on a thread so the outward heat flux to the environment is constant and is equal to heat flux given.

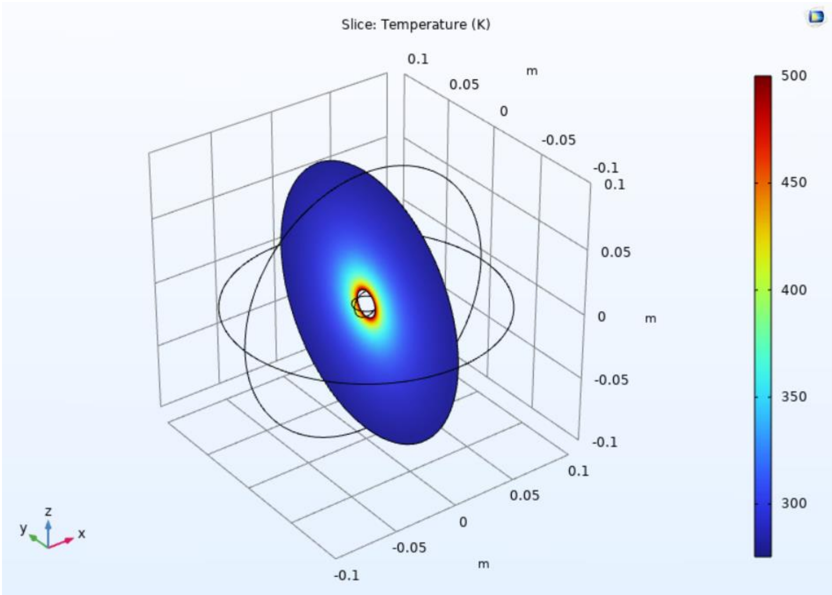


Figure 6.1: 2-D slice through the sphere. (3-D illustration)

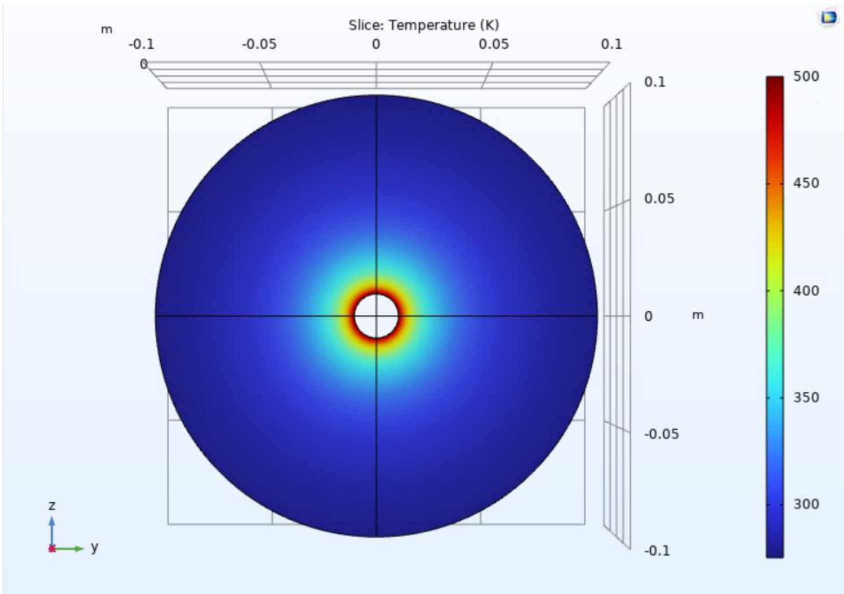


Figure 6.2: 2-D slice through the sphere.

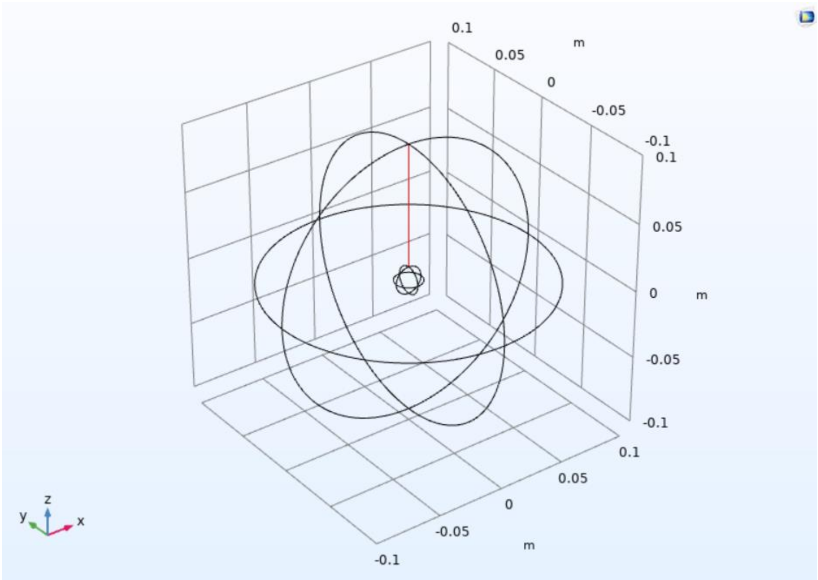


Figure 7.1: 1D plot of "T" versus "r". (Trajectory 3-D illustration)

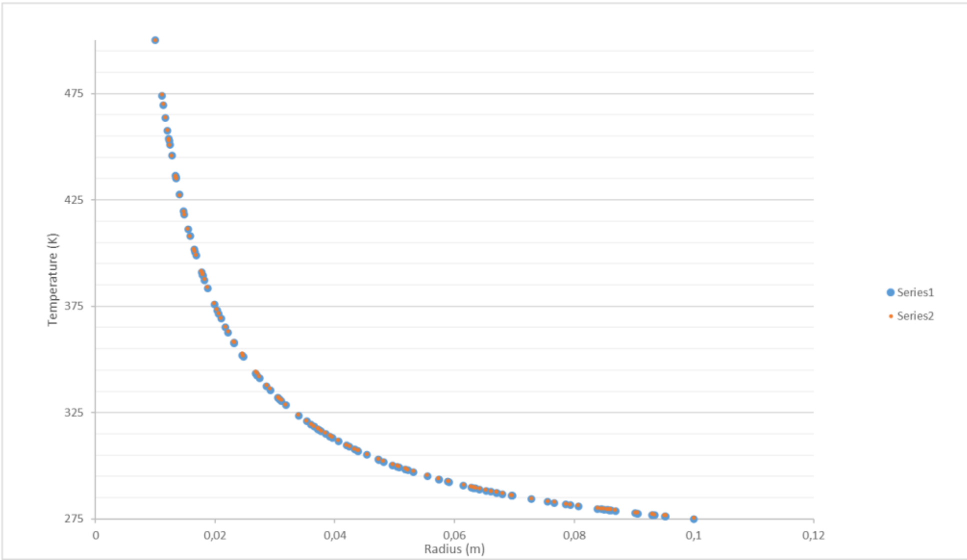
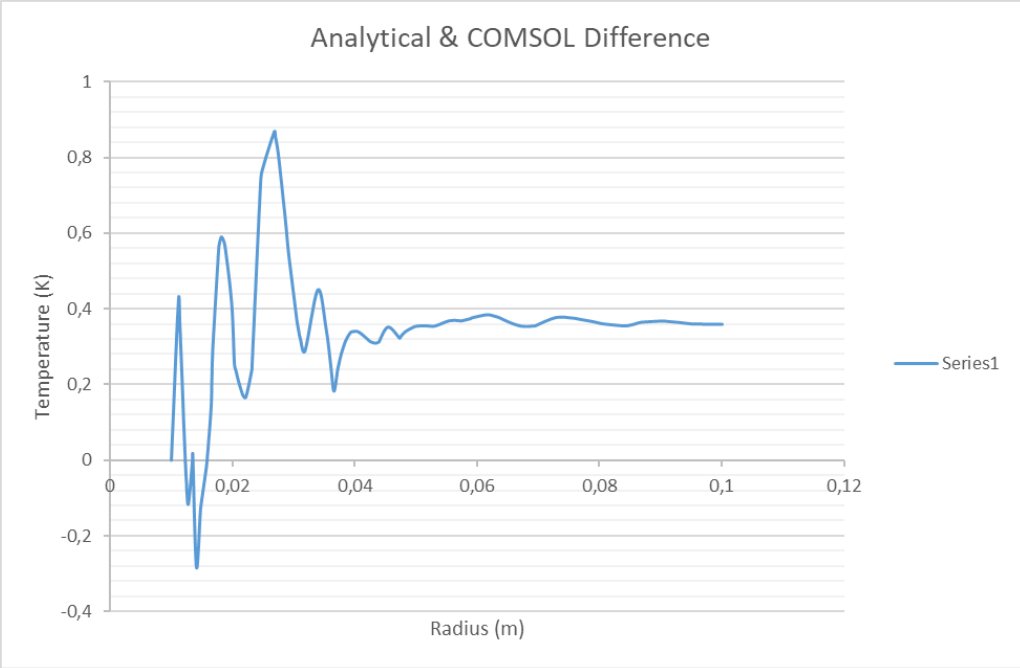


Figure 7.2: 1D plot of Temperature against radius. (Analytical in orange, COMSOL in blue)



Plot 7.3: Plot of analytical and COMSOL difference in temperature against radius

Expected same result from the analytic solution or no? Why?

Generally, the COMSOL result very closely reflects the analytical result, as 2 plots in figure 7.2 almost perfectly overlap with each other, and the difference range of these two values are between -0.3K to 0.9K, which is very small. However, there is a little difference. The difference in results in some places reached nearly 0.9K. The difference is more volatile at the beginning where the temperature value changes more rapidly. Which indicates that the simulation adaptive mesh (generates smaller 3-D shapes (for which the temperature values are calculated) to the places where the temperature value changes more rapidly and further in the places where the change is small) which decreased the uncertainty in calculation, but did not perform perfectly. This error most likely comes from analytic solution being non linear (1/x), and the COMSOL distributing the temperature between mesh points linearly.

Conclusion:

This homework exercise looked at modeling steady state conduction heat transfer through solids. Firstly, compared 1-D heat conduction in a cuboid analytic solution with COMSOL solution, and found out that they perfectly aligned each other. The error here most likely came from the amount of decimal places chosen during the calculation performance. Secondly, complex heat conduction in the cuboid with hard initial conditions and 3-D heat flows was simulated in COMSOL application and the result was beautiful. It would be very hard to derive this equation analytically thus in this case using COMSOL application served as a useful tool for finding the answer. Finally, we compared heat conduction in the sphere analytic solution with COMSOL solution, and surprisingly found out that they do not perfectly align with each other. We went on and investigated the issue and concluded that the error most likely comes from analytic solution being non linear (1/x), and the COMSOL distributing the temperature between mesh points linearly. Which was surprising.