ME 526 - Homework 3

Simulation study of transient heat conduction in a solid.

Due Date:

14th March 2023

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Introduction:

This homework exercise will try to predict the transient conduction heat transfer inside an egg which is placed from room temperature into the boiling water. The ultimate goal of exercise is to find a time for which the white of the egg will be perfectly cooked but the yolk is not (see photographs below). Two approaches will be used to find approximate solutions. First approach involves running a simulation in COMSOL. Second approach is assuming that the egg is spherical and expressing the semi analytical solution in terms of infinite series. Of course more assumptions will be used thought out the assignment. Thouse will be included in the body and conclusion of the homework.

Photo 1: Egg with perfectly cooked whites and totally not cooked yolks. (HW #3 Manual)

Part 1: Simulation with COMSOL

Assumptions:

1) **Geometry:**
    - **Ignore the shell. Egg consists of yoke and white only.**

      In reality the egg consists of three parts: the egg shell, the white albumen, and the yolk. We are going to ignore the shell as it is very thin and relatively high thermal conductivity thus will slightly influence the distribution of heat inside the egg.

- **The geometry of the egg will be assumed to be an eclipse. Which will be defined using the average egg parameters from manual.**
- **Geometry of yoke and white:**

  The yolk comprises 33% of the volume of the egg. Etc. (Look at it later)

2) **Physical and chemical thermal properties:**
   - **No Energy put in to the phase change:**

     In reality when phase change occurs the body stops heating for some time as energy goes not only in to the movement of particles but also into the creation or disctuction of new bonds between particles and into many other things. For eggs phase changes this phenomenon are very complicated in their nature because by heating egg we convert it from liquid into the solid which is weird. The only way we can consider those phenomena is by assuming that the time to cook the egg the will actually be a little longer than we predict. Thus $t_{cook,\, actual} \simeq 1.2\, t_{cook,\, calculated}$.

     This phenomena should be the greatest error cause in the experiment.
   - **Constant thermal conductivity of yoke and white before after and through out the phase change**
   - **Constant specific heat of yoke and white before after and through out the phase change**
   - **Constant density of yoke and white before after and through out the phase change:**

     This three assumptions are not perfect but somewhat valid because the range of temperatures we are dealing here is not too large
   - **The white is phase changing at 63** C°
   - **The egg yoke is transitioning at 67** C°
   - **Imidiate submerge into the water at t = 0.**
3) **Initial conditions:**
   - **At initial time for all volume inside the egg:** $T(r, \theta, \phi,\ t\ =\ 0)\ =\ T_{room}$

   We will assume initial distribution of temperature inside an egg equal to constant and equal to T_{room}. This assumption is valid if the egg is left in the room for some time (more than 10 mins should do the job).
4) **Boundary condition:**
   - **Surface temperature at:** $T_{boiling\ water}$

   In the world of heat transfer there are 3 major ways to look at boundary conditions:

   ($Bi\ = \frac{\bar{h}R}{\bar{k}}$ where: $\bar{h}$- of fluid in touch of boundary, k - the surface material parameter)

Dirchlet condition: is constant surface temp and it works well for $Bi \gg 1$
Neumann condition: constant q or insulation (Does not apply for us)

Convective condition $q_{conv} = q_{cond}$ at the boundary. Applies well for $Bi \gg 1$

For our case:

$\bar{h}_{boiling,\, water\ in\ pot} \simeq 2500 - 25000 \frac{W}{m^2 K}$ $(Ref\ 2)$

(Such big range because of the different types of bubbles)

$\bar{k}_{white} \simeq 5.4 \times 10^{-3} \frac{W}{cm\,K} \simeq 0.54 \frac{W}{m\,K}$

$R \simeq 0.05m$

Thus:

$Bi \simeq 231.5 - 2314.8$

$Bi \gg 1$

This implies that actually the temperature distribution will be mostly around the body and not inside it. However, for the purpose of simplicity, we will assume the surface temperature to be $T_{boiling\ water}$. This assumption will cause a lot of error in our real-world implication.

Variables used for simulation:

- **Geometry variables:**

Lets talk geometry:

elipce 1:

$$\boxed{\begin{array}{lll} w_x = 4.2\,cm & = 0.042\,m & (Manual) \\ w_y = 4.2\,cm & = 0.042\,m & (Manual) \\ h = 5.6\,cm & = 0.056\,m & (Manual) \end{array}}$$

$V_{total} = \frac{4}{3}\pi \cdot 0.042^2 \times 0.056$

$\frac{1}{3} V_{yoke} = V_{total}$

For yoke same shape as egg located at the center:

$$\cancel{\tfrac{4}{3}}\pi\left(\overset{(\frac{3}{4})}{\frac{0.042}{0.056}}\, d\right)^2 d = \overset{\frac{1}{3}}{\cancel{\tfrac{4}{3}}}\pi \cdot 0.042^2 \times 0.056$$

$\frac{9}{16} d^3 = 0.000033$

$d = 0.03883\,m$

$\frac{3}{4} d = 0.02912\,m$

elipce 2:

$$\boxed{\begin{array}{l} w_x = 0.029\,m \\ w_y = 0.029\,m \\ h = 0.039\,m \end{array}}$$

Figure 1: Derivation of second eclipse dimensions

The photo above illustrates how we acquired dimensions for the second (yoke) eclipse.

$$w_{x,1} = 0.042\, m \ ; \ w_{y,1} = 0.042\, m \ ; \ h_{z,1} = 0.056\, m$$

$$w_{x,2} = 0.029\, m \ ; \ w_{y,2} = 0.029\, m \ ; \ h_{z,2} = 0.039\, m$$

And be the form of the eclipse:

- Physical

$$c_{Yolk} = 2.7\, \frac{J}{gK} \ ; \ K_{Yolk} = 0.0034\, \frac{W}{cmK} \ ; \ \rho_{Yolk} = 1.032\, \frac{g}{cm^3}$$

$$c_{White} = 3.7\, \frac{J}{gK} \ ; \ K_{White} = 0.0054\, \frac{W}{cmK} \ ; \ \rho_{White} = 1.038\, \frac{g}{cm^3}$$

Convert to SI units:

$$c_{Yolk} = 2700\, \frac{J}{kgK} \ ; \ K_{Yolk} = 0.34\, \frac{W}{mK} \ ; \ \rho_{Yolk} = 1032\, \frac{kg}{m^3}$$

$$c_{White} = 3700\, \frac{J}{kgK} \ ; \ K_{White} = 0.54\, \frac{W}{mK} \ ; \ \rho_{White} = 1038\, \frac{kg}{m^3}$$

- Thermal

$$T_{initial} = 292,15\, K \ ; \ T_{surface} = 373,15\, K$$

Create the thirteen curves mentioned above and plot them versus each other:

Curves at t = 0; 30; 60; 90; 120; 150; 180; 210; 240; 270; 300; 330; 360 seconds, that represent the temperature distribution inside the solid.

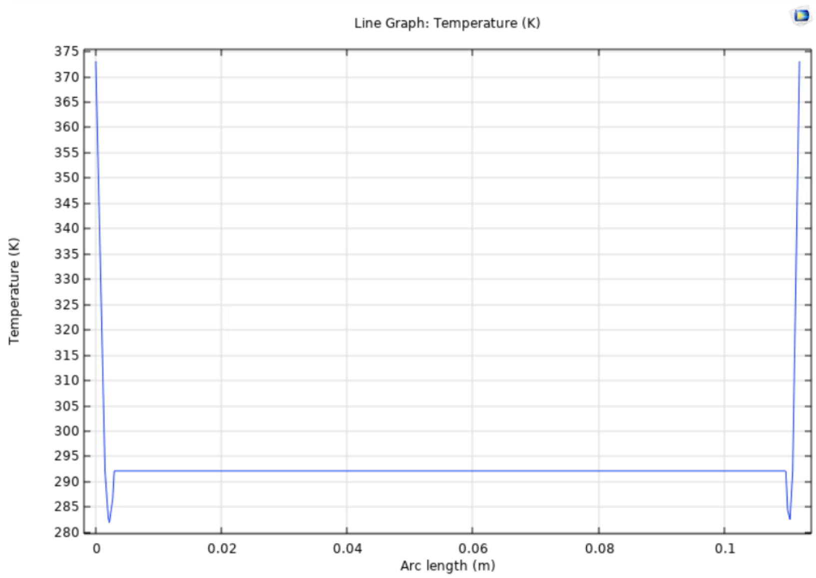

Figure 2: distribution of temperature for center axis (y- axis) line of eclipse for time: 0 s

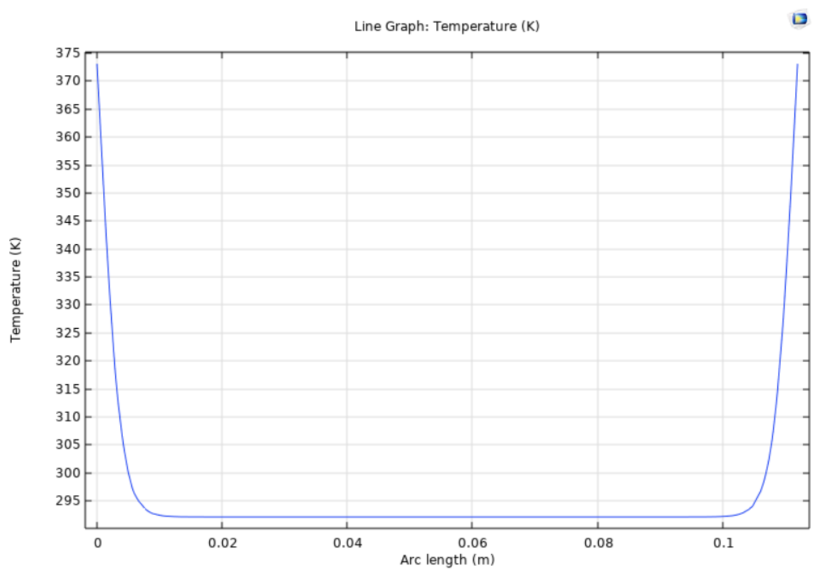

Figure 3: distribution of temperature for center axis (y- axis) line of eclipse for time: 30 s

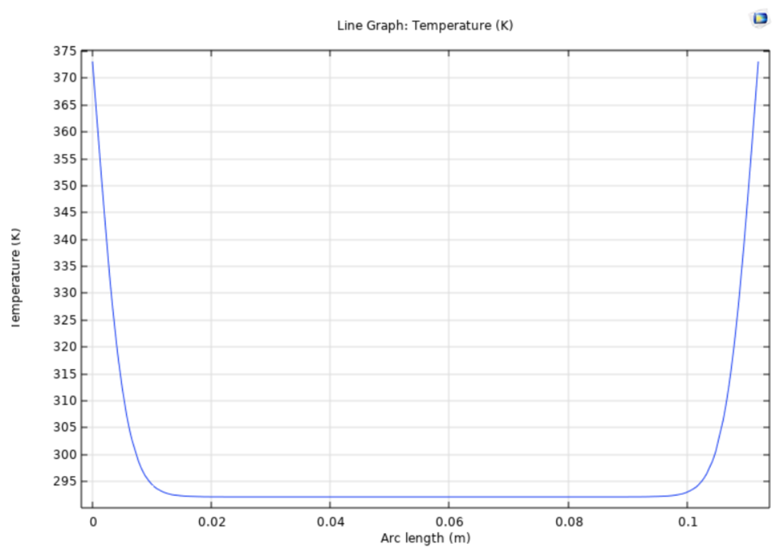

Figure 4: distribution of temperature for center axis (y- axis) line of eclipse for time: 60 s

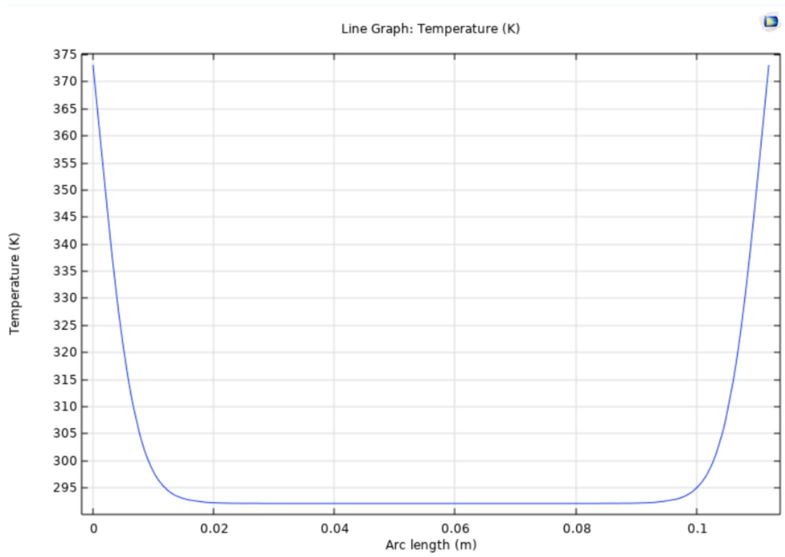

Figure 5: distribution of temperature for center axis (y- axis) line of eclipse for time: 90 s

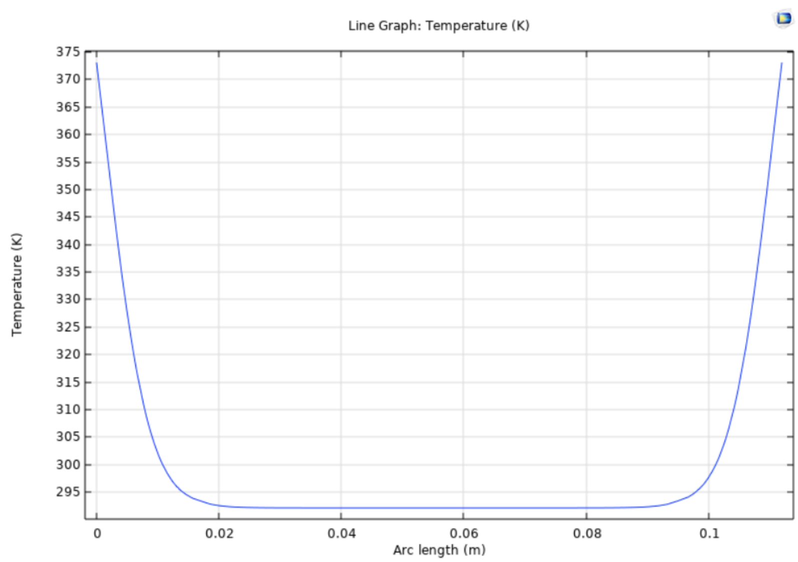

Figure 6: distribution of temperature for center axis (y- axis) line of eclipse for time: 120 s

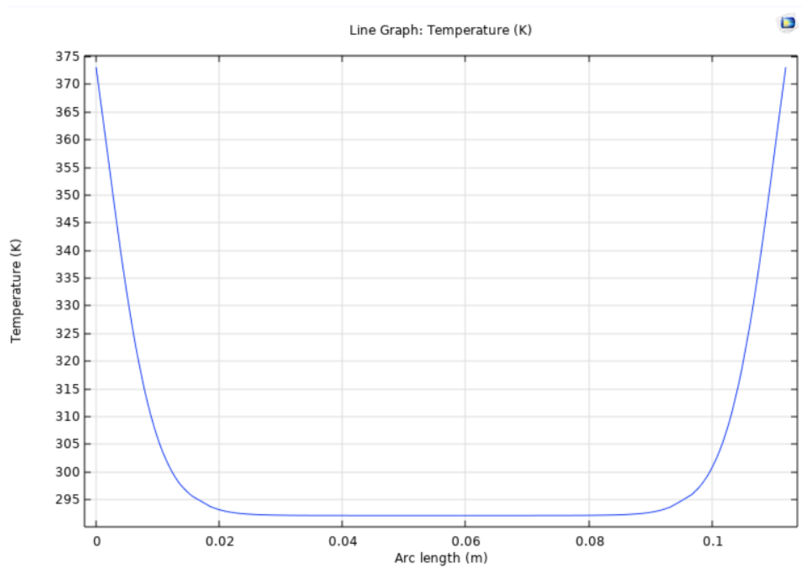

Figure 7: distribution of temperature for center axis (y- axis) line of eclipse for time: 150 s

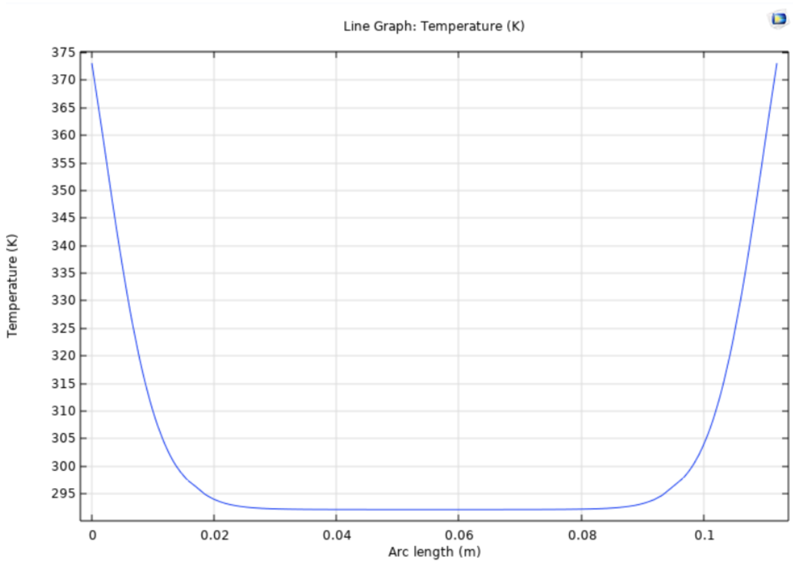

Figure 8: distribution of temperature for center axis (y- axis) line of eclipse for time: 180 s

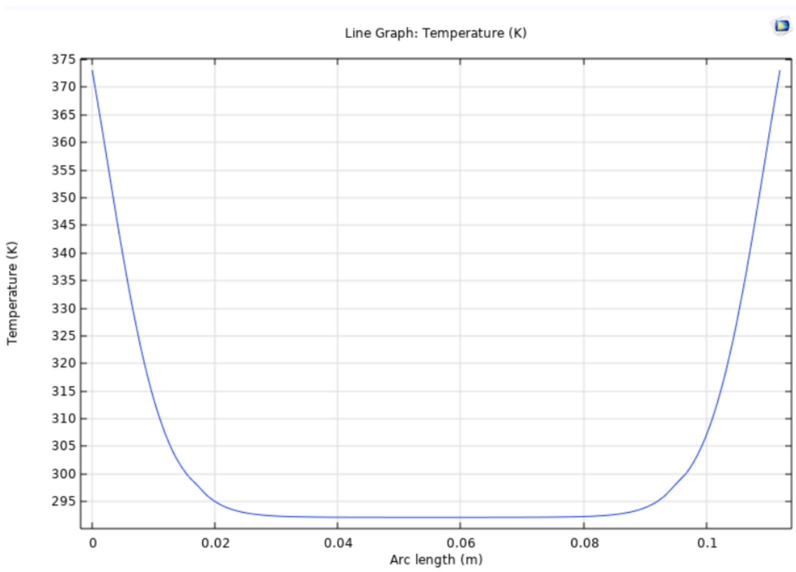

Figure 9: distribution of temperature for center axis (y- axis) line of eclipse for time: 210 s

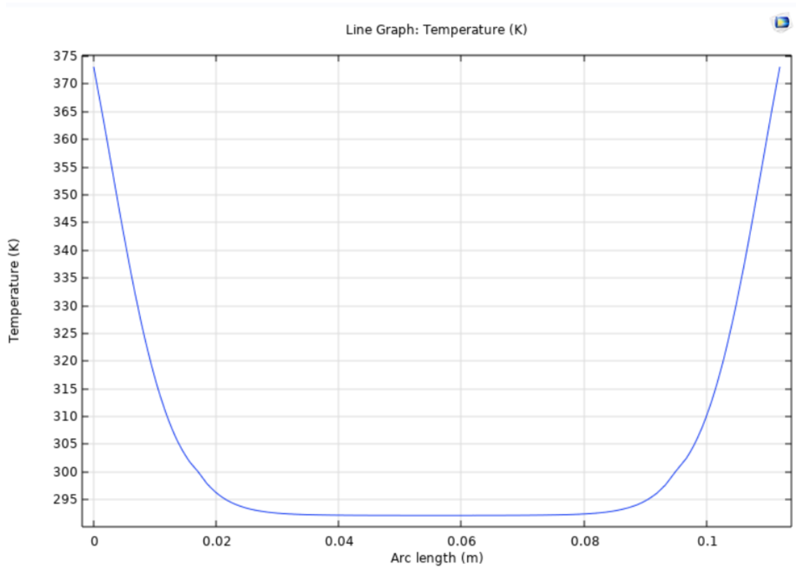

Figure 10: distribution of temperature for center axis (y- axis) line of eclipse for time: 240 s

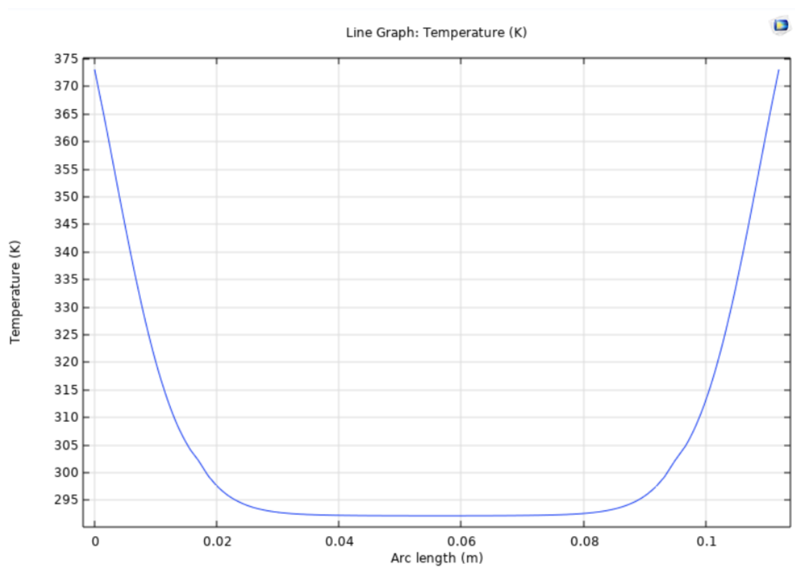

Figure 11: distribution of temperature for center axis (y- axis) line of eclipse for time: 270 s

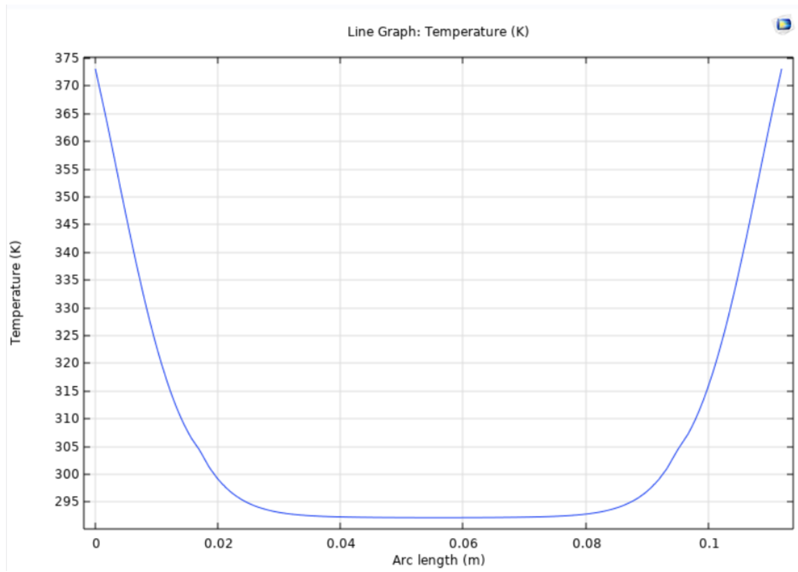

Figure 12: distribution of temperature for center axis (y- axis) line of eclipse for time: 300 s

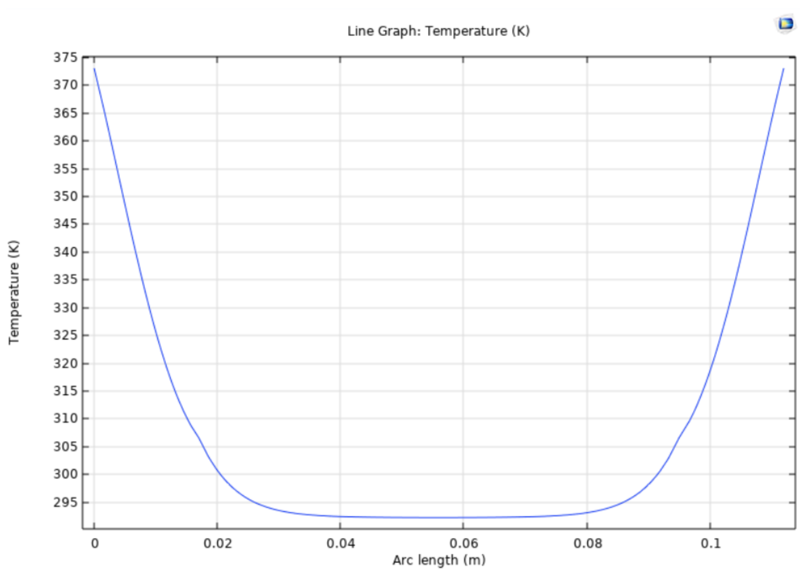

Figure 13: distribution of temperature for center axis (y- axis) line of eclipse for time: 330 s

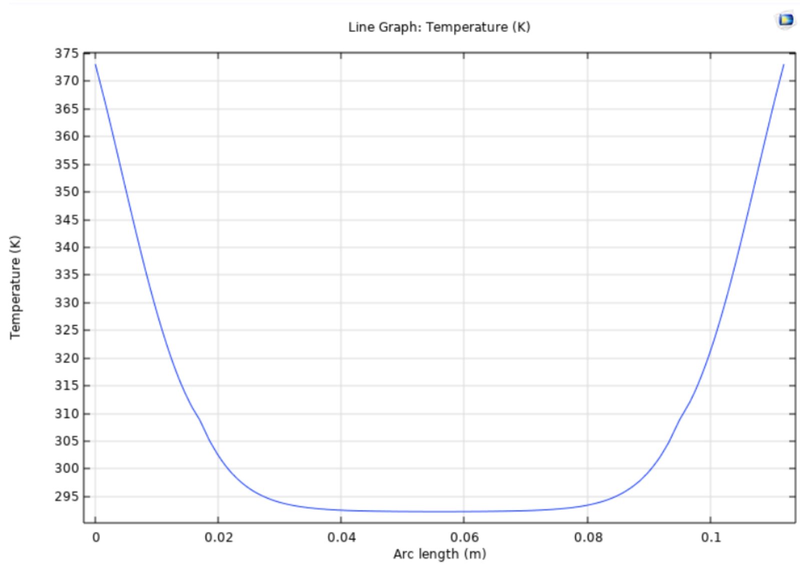

Figure 14: distribution of temperature for center axis (y- axis) line of eclipse for time: 360 s

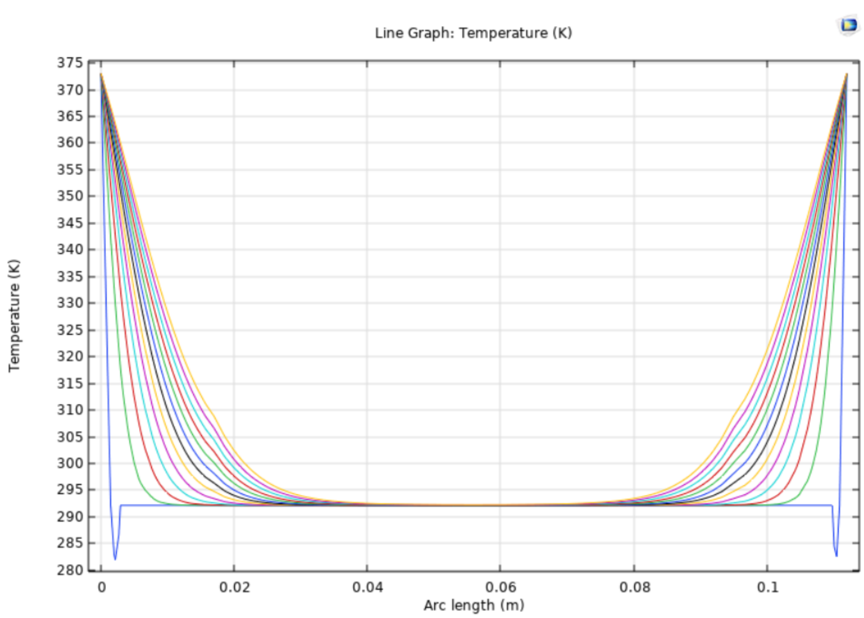

Figure 15: distribution of temperature for center axis (y- axis) line of eclipse for all times above on one plot!

The last plot clearly indicates that the temperature distribution inside the ellipse is slightly creeping inside the egg which perfectly aligns with logic.

Create seven 2-D color slices through the egg at times t = 0; 60; 120; 180; 240; 300; 360 seconds.

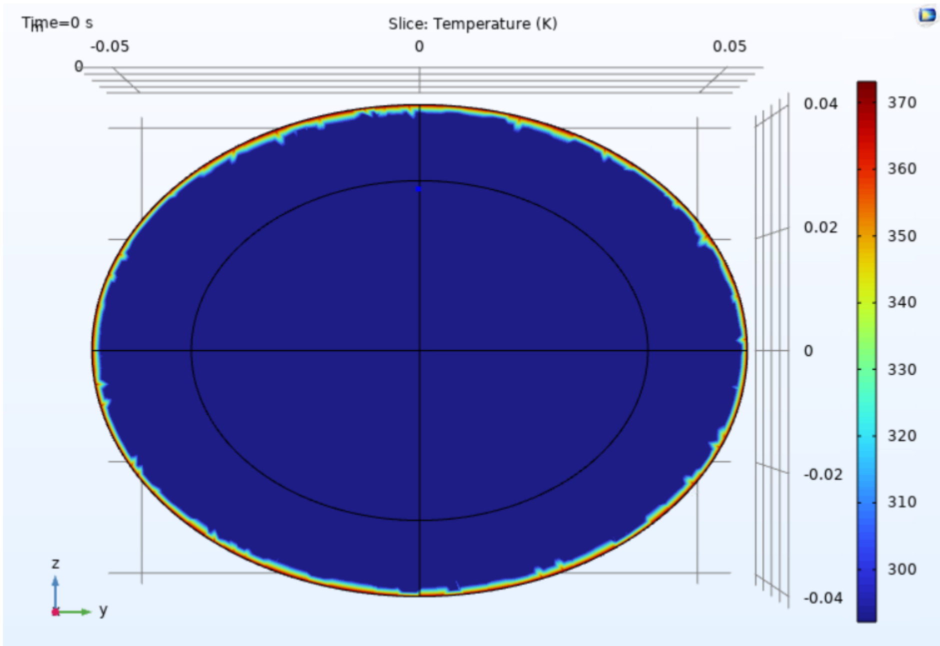

Figure 16: distribution of temperature for 2-D YZ plane of eclipse for time = 0 s

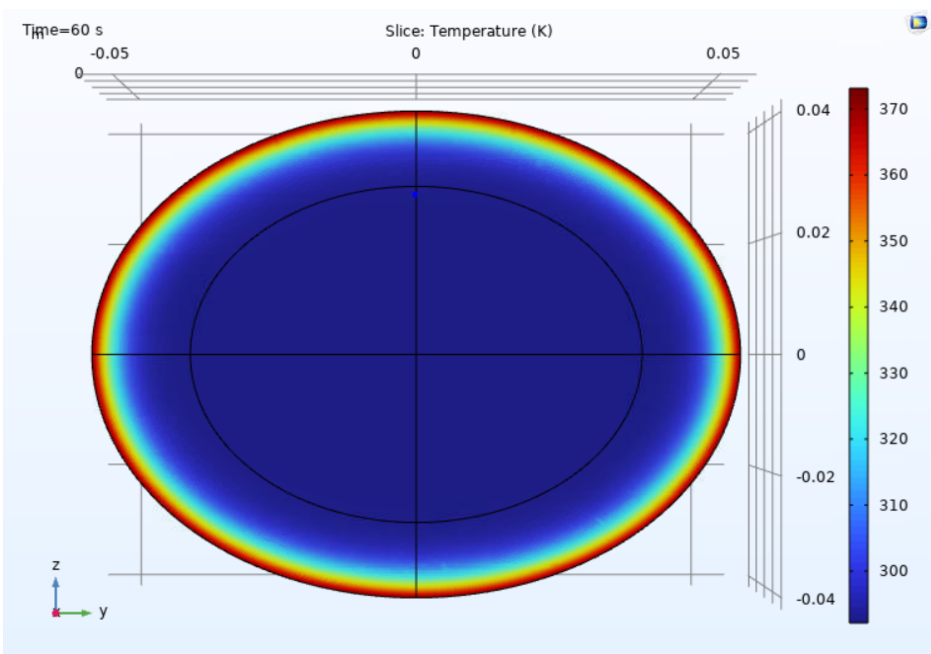

Figure 17: distribution of temperature for 2-D YZ plane of eclipse for time = 60 s

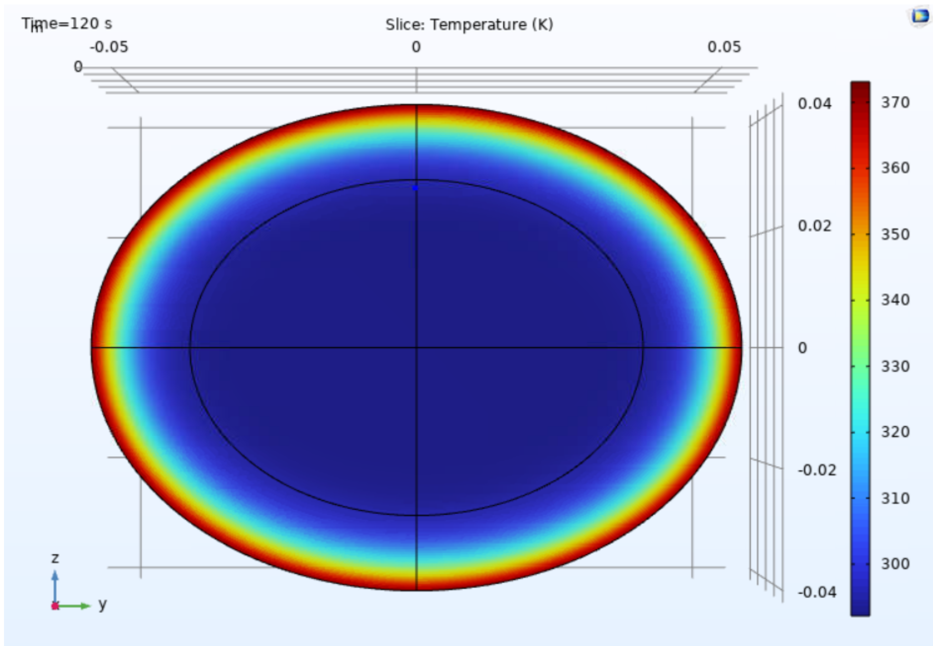

Figure 18: distribution of temperature for 2-D YZ plane of eclipse for time = 120 s

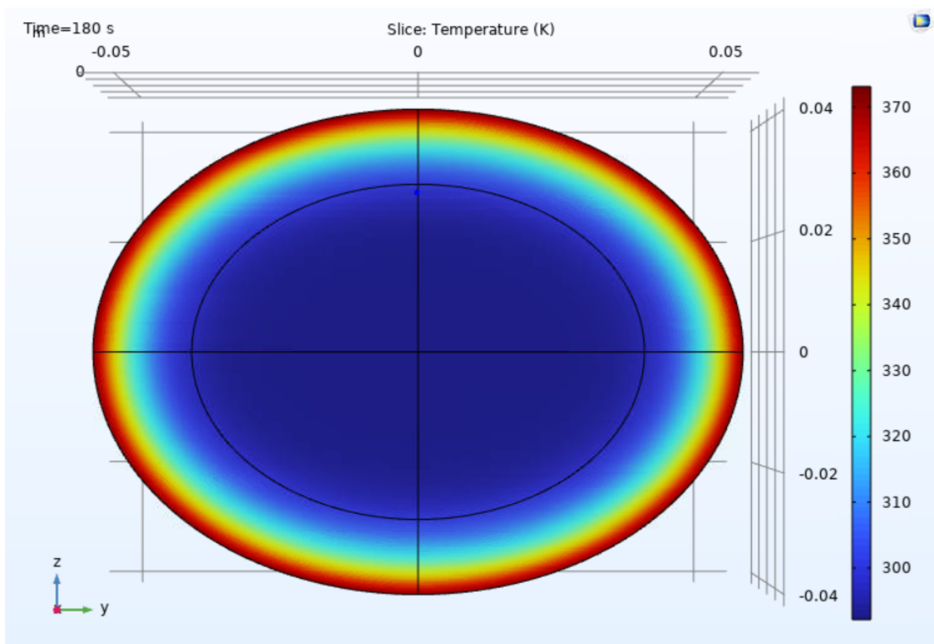

Figure 19: distribution of temperature for 2-D YZ plane of eclipse for time = 180 s

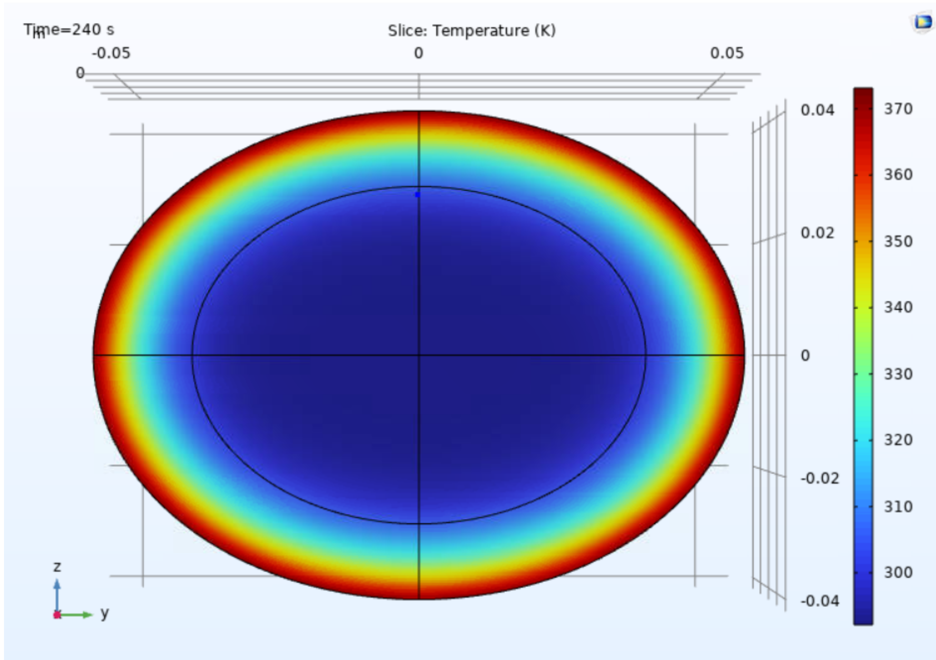

Figure 20: distribution of temperature for 2-D YZ plane of eclipse for time = 240 s

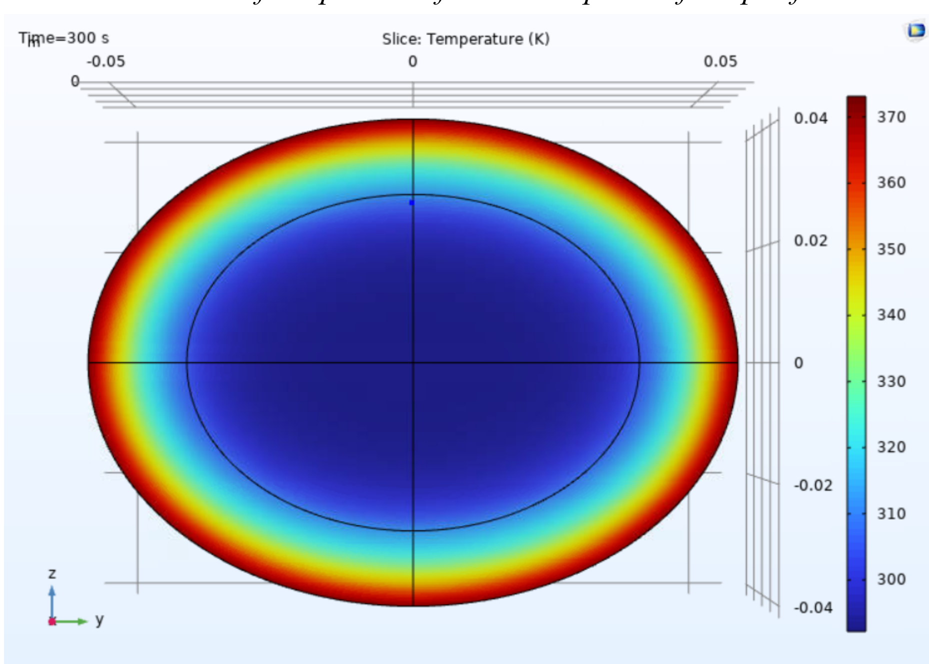

Figure 21: distribution of temperature for 2-D YZ plane of eclipse for time = 300 s

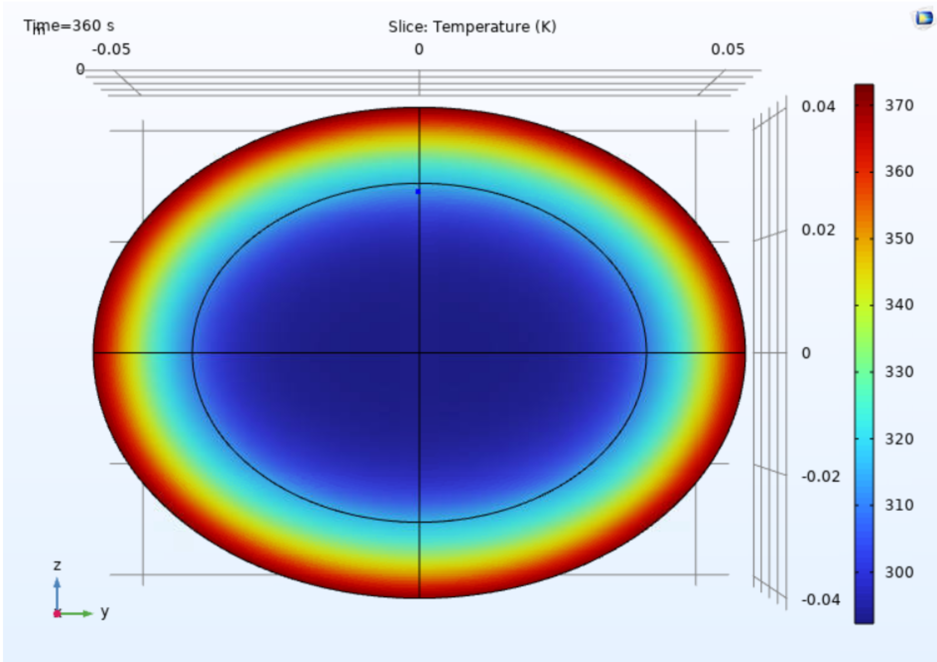

Figure 22: distribution of temperature for 2-D YZ plane of eclipse for time = 360 s

Part 2: Semi-analytic result

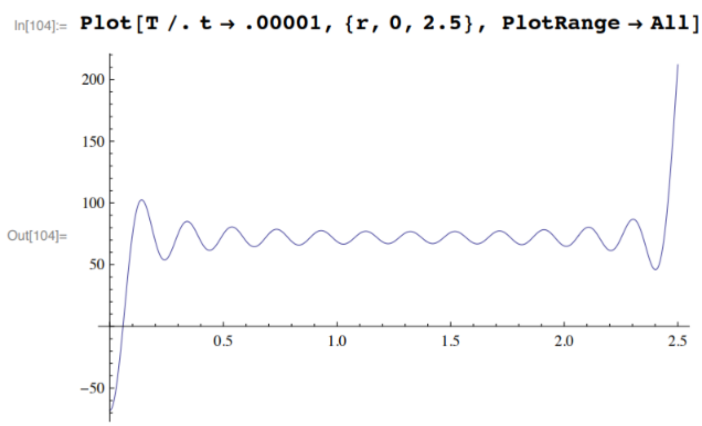

Figure 23: distribution of temperature for center axis (y- axis) line of eclipse for time: 0 s

For the first plot here, 25 terms are shown here at time 0.00001 ~ 0 from a radius of 0 to 2.5. This inherently assumes that the egg has a total spherical diameter of 5. On the right, the temperature is 212 Fahrenheit - because the egg on the very surface of it is equal to the boiling water at 212 Fahrenheit.

Two things are to be noted here - the first is that there is a discontinuity on both extreme ends of the graph. This is due mainly to the Fourier Series having a large error near discontinuities such as the end of the egg and middle of the egg. This error is known as *Gibbs* *phenomeno*n, which is the continuous overshoot or undershoot of the values of a partial sum expansion of a function near a jump discontinuity in comparison to the values of the original function. The Nth partial Fourier series of the function, which is created by adding together its N lowest constituent sinusoids, results in a big peak surrounding the jump that overshoots and undershoots the function's true values.

It is also important to note that since this graph is 0 -> 2.5, it does not consider the other side of the egg from r = 0 to -2.5, but since this is a sphere, it will be a mirror image of the graph.

```
Plot[T /. t → .00001, {r, 0, 2.5}, PlotRange → All]
```

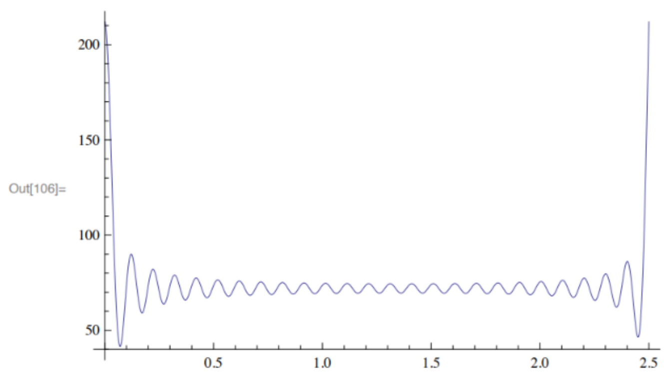

Figure 24: distribution of temperature for center axis (y- axis) line of eclipse for time: 0 s

```
Plot[T /. t → 0.00001, {r, 0, 2.5}, PlotRange → All]
```

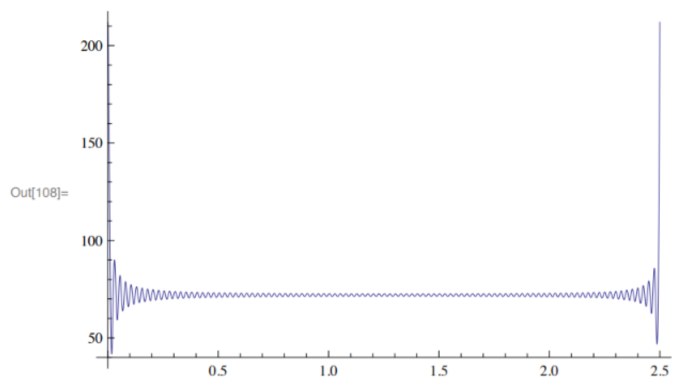

Figure 25: distribution of temperature for center axis (y- axis) line of eclipse for time: 0 s

Here, we have increased the number of terms to 50 and 100. We can see that the line is starting to get flat here, but still with errors at both extreme ends where there is discontinuation due to *Gibbs phenomeno*n.

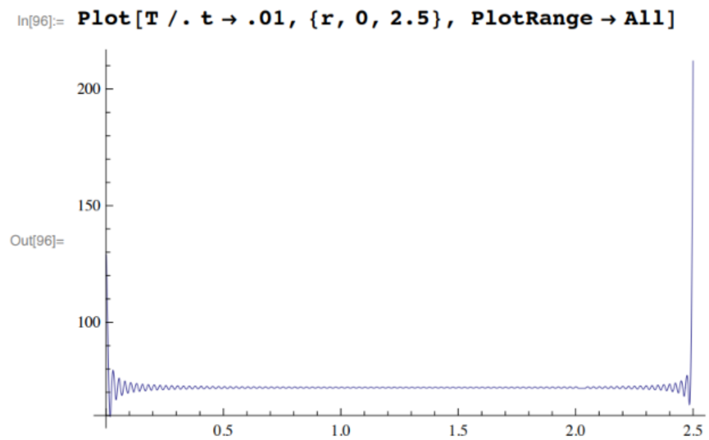

Figure 26: distribution of temperature for center axis (y- axis) line of eclipse for time: 0.01 s

Next, we plot 200 points, but across a time period of 0.01. It is interesting to note that on top of forming an even straighter line (less visible oscillations), the error on the left has significantly decreased.

The estimated answer becomes more exact as the time and number of terms analyzed grow. This is due to the fact that a greater number of terms in the series may capture more of the subtle characteristics of the temperature distribution, resulting in a smoother curve. Furthermore, as time passes, the temperature distribution inside the egg varies, causing the precise solution to alter. This change is therefore captured and can be used to achieve a more accurate depiction of the temperature distribution by assessing the series for a longer period of time. This is illustrated by the following few graphs below as well.

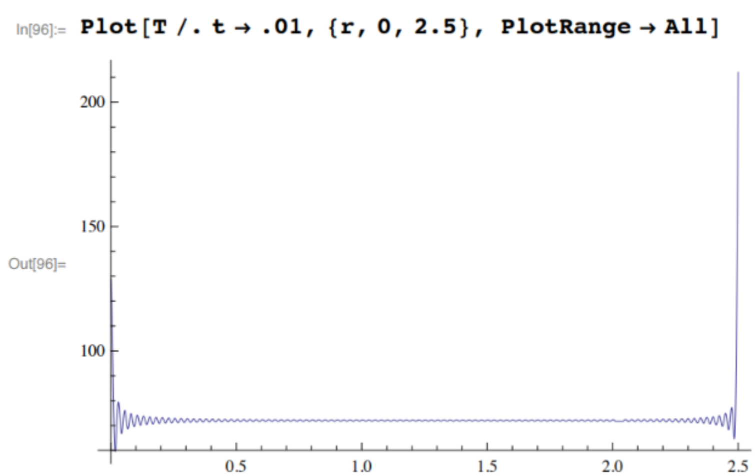

Figure 27: distribution of temperature for center axis (y- axis) line of eclipse for time: 0.01 s

```
In[97]:= Plot[T /. t → .1, {r, 0, 2.5}, PlotRange → All]
```

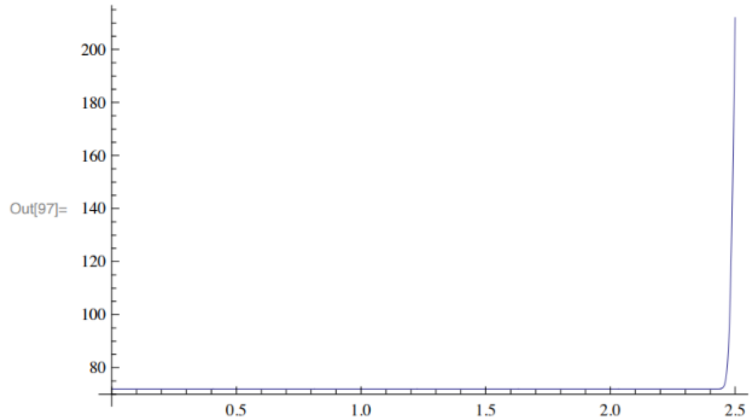

Figure 28: distribution of temperature for center axis (y- axis) line of eclipse for time: 0.1 s

```
In[98]:= Plot[T /. t → 1.0, {r, 0, 2.5}, PlotRange → All]
```

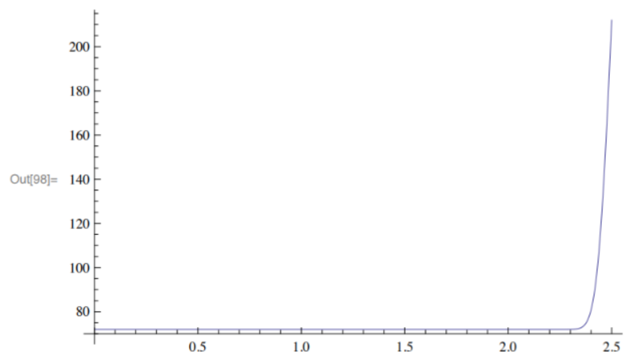

Figure 29: distribution of temperature for center axis (y- axis) line of eclipse for time: 1 s

```
In[99]:= Plot[T /. t → 10, {r, 0, 2.5}, PlotRange → All]
```

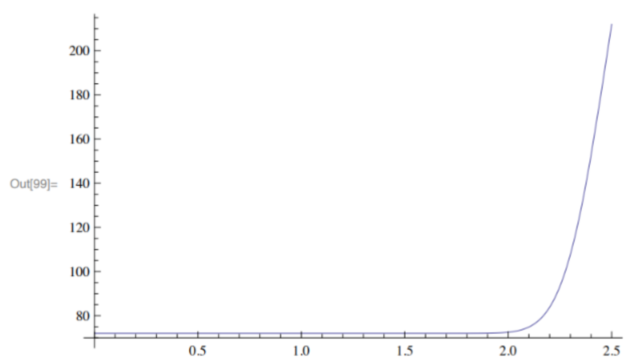

Figure 30: distribution of temperature for center axis (y- axis) line of eclipse for time: 10 s

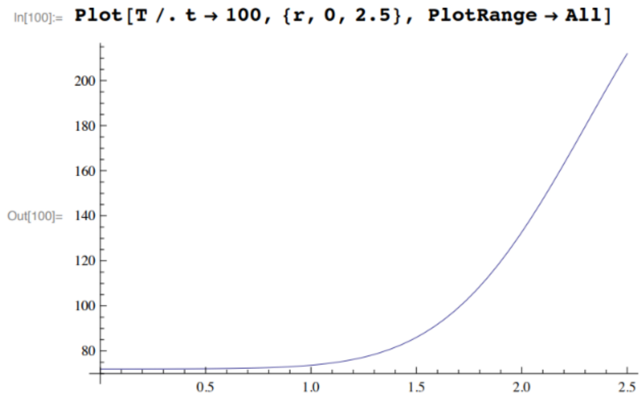

Figure 31: distribution of temperature for center axis (y- axis) line of eclipse for time: 100 s

For next few graphs below, only one term is used. We observe that the curves are more smooth compared to previous plots, but start in the negative range (error). This is because when using only one term from the infinite series, you are effectively using a single exponential function to approximate the temperature distribution inside the egg. Because this exponential function is centered on the origin (i.e., the center of the egg), it represents the main behavior of the temperature distribution towards the center.

This approximation, may not be true for places farthest from the center, where the temperature distribution may change. Moreover, the exponential function can assume negative values at the origin, which may be physically meaningless. This is due to the fact that the temperature inside the egg cannot reach negative.

Therefore, while using a single term may result in a smoother curve close to the center of the egg, it can lead to inaccuracies in the temperature distribution at other regions and produce negative values close to the origin. By including more terms in the infinite series, we can capture more of the intricacies of the temperature distribution and obtain a more accurate approximation without negative values.

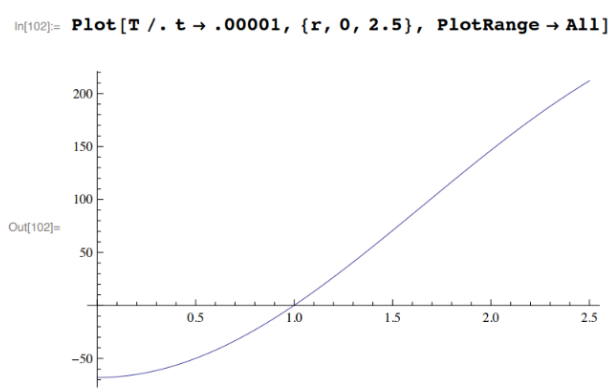

Figure 32: distribution of temperature for center axis (y- axis) line of eclipse for time: 0 s

```
In[110]:= Plot[T /. t → .01, {r, 0, 2.5}, PlotRange → All]
```

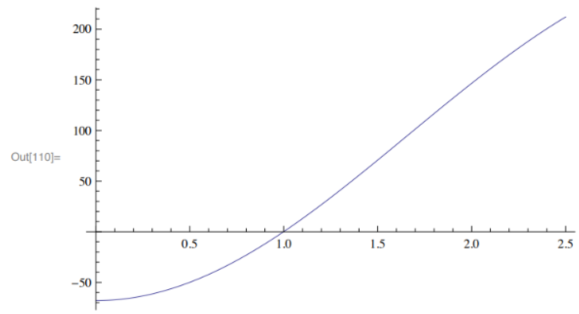

Figure 33: distribution of temperature for center axis (y- axis) line of eclipse for time: 0.01 s

```
In[111]:=
Plot[T /. t → .1, {r, 0, 2.5}, PlotRange → All]
```

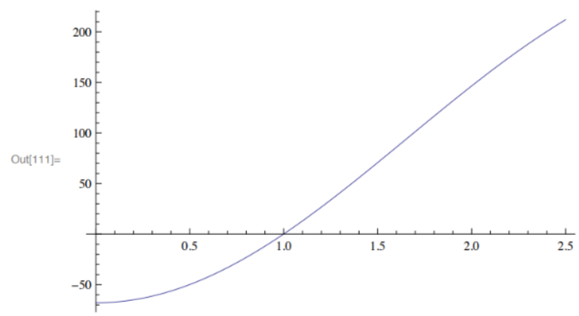

Figure 34: distribution of temperature for center axis (y- axis) line of eclipse for time: 0.1 s

```
In[112]:= Plot[T /. t → 1.0, {r, 0, 2.5}, PlotRange → All]
```

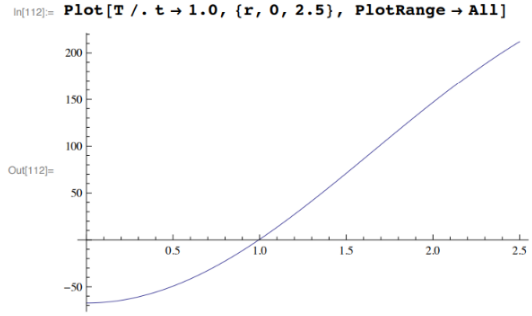

Figure 35: distribution of temperature for center axis (y- axis) line of eclipse for time: 1 s

In[113]:=

```
Plot[T /. t → 10, {r, 0, 2.5}, PlotRange → All]
```

Out[113]=

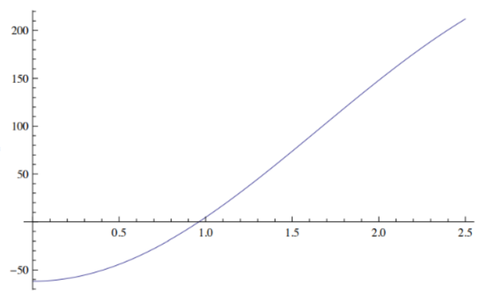

Figure 36: distribution of temperature for center axis (y- axis) line of eclipse for time: 10 s

In[114]:=

```
Plot[T /. t → 100, {r, 0, 2.5}, PlotRange → All]
```

Out[114]=

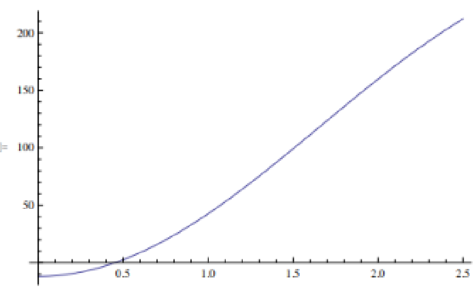

Figure 37: distribution of temperature for center axis (y- axis) line of eclipse for time: 100 s

In[116]:=

```
Plot[T /. t → 300, {r, 0, 2.5}, PlotRange → All]
```

Out[116]=

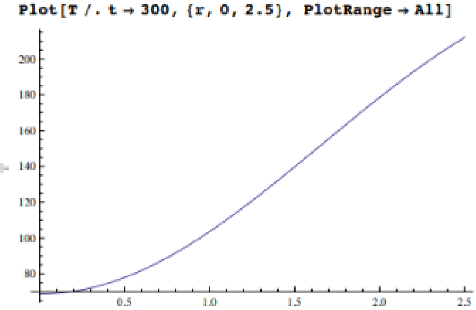

Figure 38: distribution of temperature for center axis (y- axis) line of eclipse for time: 300 s

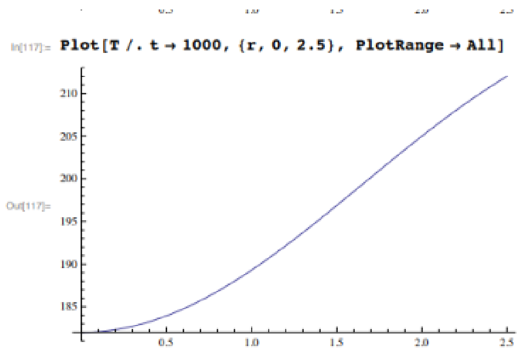

Figure 39: distribution of temperature for center axis (y- axis) line of eclipse for time: 1000 s

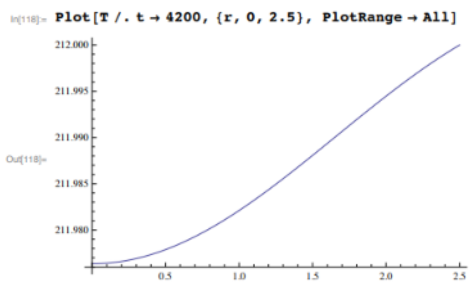

Figure 40: distribution of temperature for center axis (y- axis) line of eclipse for time: 4200 s

In contrast, on the last ones above, if we use 200 terms:

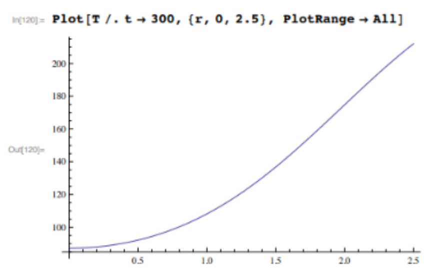

Figure 41: distribution of temperature for center axis (y- axis) line of eclipse for time: 300 s (200 terms)

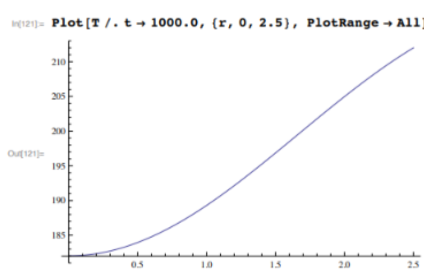

Figure 42: distribution of temperature for center axis (y- axis) line of eclipse for time: 1000 s (200 terms)

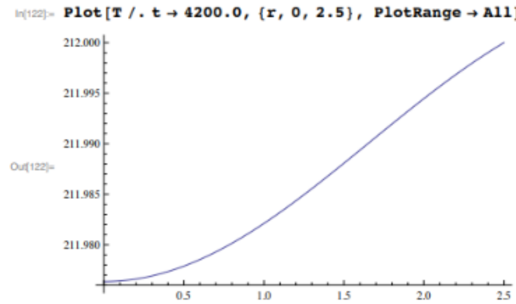

Figure 43: distribution of temperature for center axis (y- axis) line of eclipse for time: 4200 s (200 terms)

Part 3: Predicting when only the egg white is cooked

Let's finally answer the question of at what time will the egg white will be cooked but the egg yoke will stay perfectly not cooked. For that question we need to find the point located on the surface of egg yoke for which the shortest distance from edge of egg to egg yoke occurs. (we use eclipse in eclipse thus for us all the distances are the same) Then find time at which the temperature in this point is 67 C°(340,15 K).

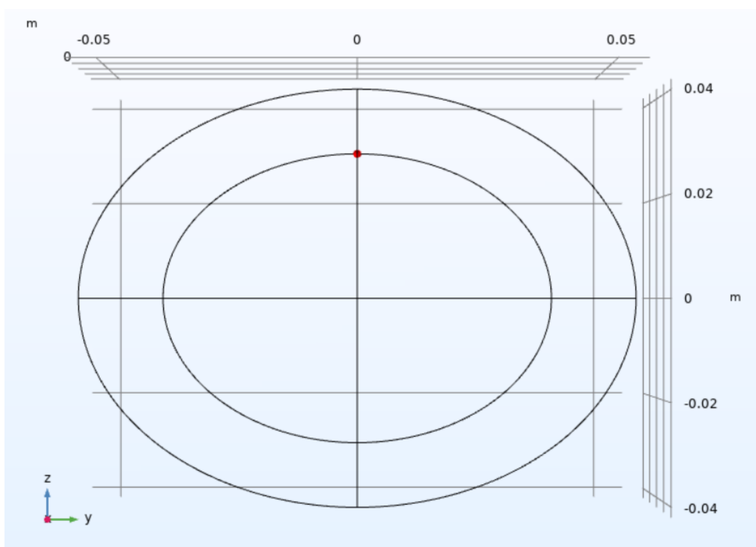

Figure 44: Illustration of location of point (in red)

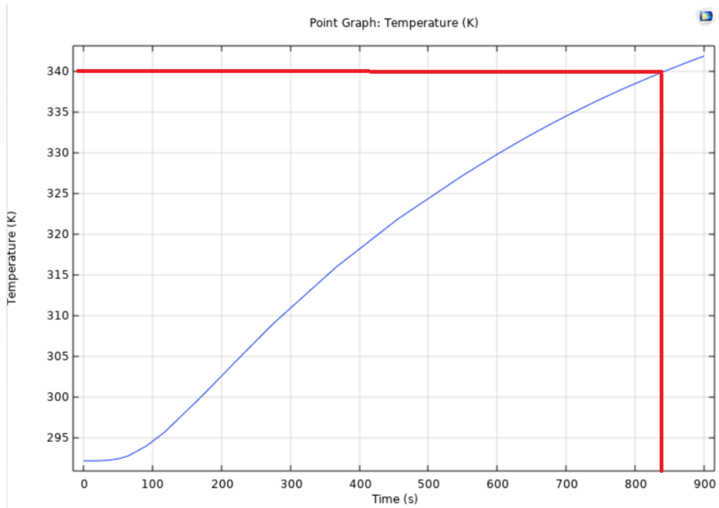

Figure 45: Temperature vs Time at the point from Figure 44

The figure above indicates that the time at which the egg will perfectly cook it's white and not touch its yoke is approximately 835 seconds. This time the result does not reflect on many physical realities (the largest source of uncertainty might be around ≃±0.3 result magnitude error) also the result includes all the computational errors the COMSOL software produces (minor ≃±0.5 K error).

Conclusion:

In conclusion, this homework exercise predicted the transient conduction heat transfer inside an egg that is placed from room temperature into boiling water. We brainstormed the different assumptions we make and how the effect resulted. Then we generated the best possible geometric & physical model given the time we had and software and knowledge. The next step was running the COMSOL simulation and generating plots along the center axis, together with the 2-D plane cuts at particular times. Then we used a semi-analytic solution provided by the professor to plot the temperature distribution along the center axis of the egg. And finally, we attempted to predict the time it takes to perfectly cook the egg whites and not the egg yokes.