ME 526 - Homework 4

ELECTROSTATICS

Due Date:

28th March 2023

(Group № 2)

Member №1: Gabriel Goh

Member №2: Amirreza Rezaeepazhand

Member №3: Naiwen Hu

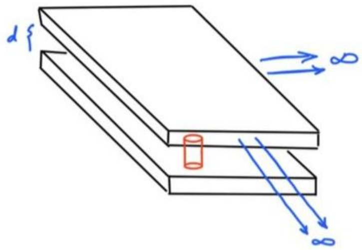
Member №4: Yury Luzhkov

Introduction:

This homework looks at how a simple capacitor field looks in different dimensions. First we derive an analytic solution for the capacitance of a parallel plate capacitor starting with Maxwell solution $\nabla \cdot D = \rho$. We also built a 3D and 2D electrostatic simulation model separately, producing the mesh image, potential vs position graph to demonstrate the electric field with contour lines. After that, we found the capacitance via Q/V for 3D and 2D cases and compared it to an analytical solution to find the most accurate solution.

Part 1: Analytic derivation for the capacitance of a parallel plate capacitor

Assuming a pair of infinite plates with distance «d» between them.



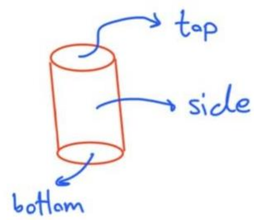
Also, assuming Gaussian pillbox between plates. (red part in the figure)

Assuming macroscopic electric field in a medium (D) and Charge density (ρ_{free}), we would have: $\nabla \cdot \vec{D} = \rho_{free}(\vec{x})$

Integrating over pillbox: $\int_{V_{pillbox}} dV \, \nabla \cdot \vec{D} = \int_{V_{pillbox}} dV \, \rho_{free}(\vec{x})$

Rewriting LHS with divergence theorem $\left(\int_{V_{total}} dV \, \nabla \cdot \vec{D} = \int_{S_{total}} ds \, \vec{n}(\vec{x}) \cdot \vec{D}(\vec{x}) \right)$:

$$\Rightarrow \int_{S_{total}} ds \, \vec{n}(\vec{x}) \cdot \vec{D}(\vec{x}) = \int_{V_{pillbox}} dV \, \rho_{free}(\vec{x})$$



$$\int_{S_{top}} ds \, \hat{n} \cdot \vec{D} + \int_{S_{bottom}} ds \, \hat{n} \cdot \vec{D} + \int_{side} ds \, \hat{n} \cdot \vec{D}$$

↳ this term (S_{top}) going to be zero, because of steady state $\vec{D} = 0$ on top

→ this term (side) also going to be zero, because $\hat{n}$ is horizontal and $\vec{D}$ is vertical, so $\hat{n} \cdot \vec{D} = 0$

$$\Rightarrow \int_{S_{bottom}} ds \, \hat{n} \cdot \vec{D} = \int_{V_{pillbox}} dV \, \rho_{free}(\vec{x})$$

D is constant on the bottom surface and in the same direction with $\hat{n}$. So, $\int_{S_{bottom}} ds\, \hat{n}\cdot\vec{D} = |\vec{D}|A$

Also the $\int_{V_{pill\,box}} dV\, \rho_{free}(\vec{x})$ is equal to charge "Q_A" inside the pill box

$$\left.\begin{array}{l} \Rightarrow |\vec{D}|A = Q_A \\ \text{Gauss's law: } \vec{D} = \epsilon\vec{E} \end{array}\right\} \Rightarrow \epsilon|\vec{E}|A = Q_A$$

which $\epsilon = \epsilon_r \epsilon_0$, $\epsilon_r = 6$, $\epsilon_0 = 8.854\times10^{-12}\ F/m$

Also, we know $|\vec{E}| = \dfrac{V}{d}$

And finally we have: $\epsilon \dfrac{V}{d} A = Q_A \Rightarrow Q_A = \underbrace{\left(\dfrac{\epsilon A}{d}\right)}_{C} V$

$$\Rightarrow \boxed{C = \frac{\epsilon A}{d}}$$

In here we assumed charge inside pillbox $= Q_A$, so the A in the formula is the bottom (or top) surface of the pillbox.

Part 2: 3D electrostatic simulation

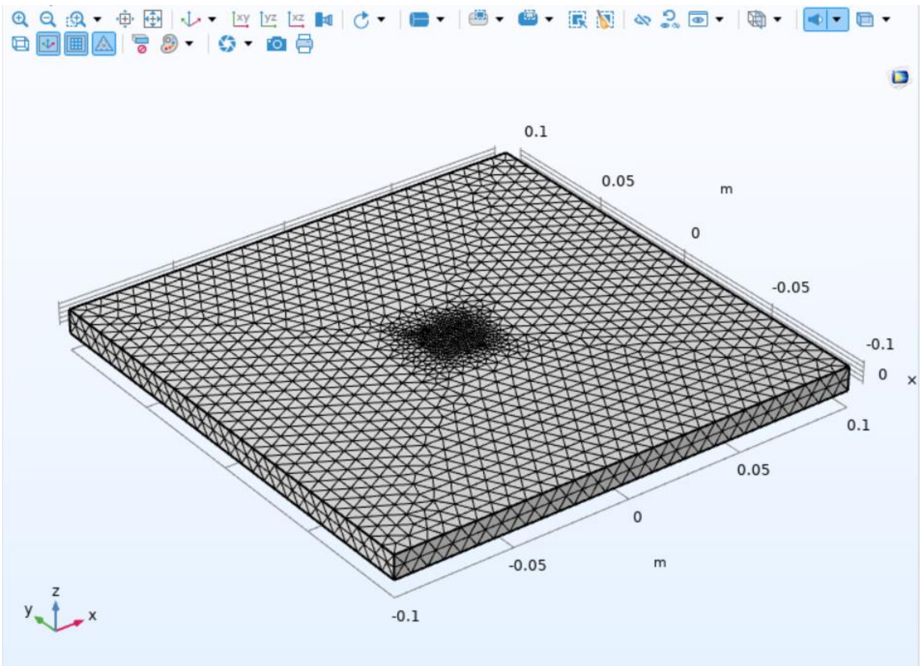


Fig.1: Full view of computational mesh.

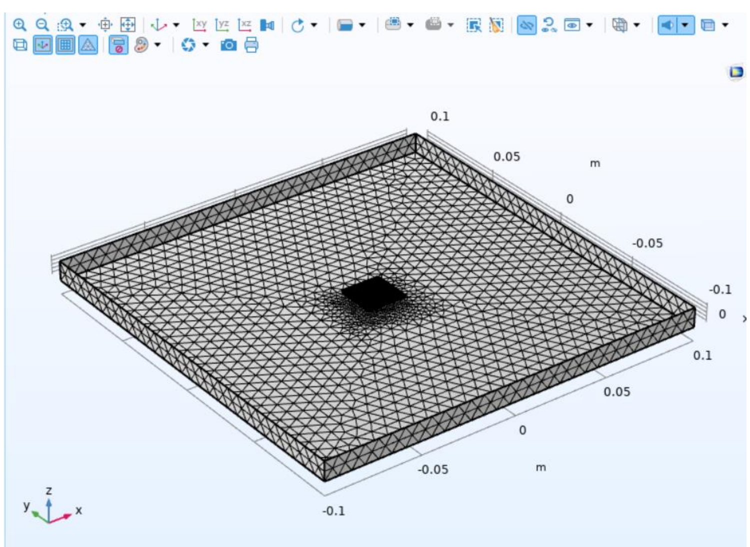


Fig. 2: Computational mesh with the top made invisible.

Why has the mesh been made this way? Clearly, Fig. 1 does not look like it has 500K elements.

The more mesh elements, the more accurate the simulation of the real system, but it will also consume more time to do simulation. For this, they refine the mesh structure less than 500k to optimize for less time cost and less error.

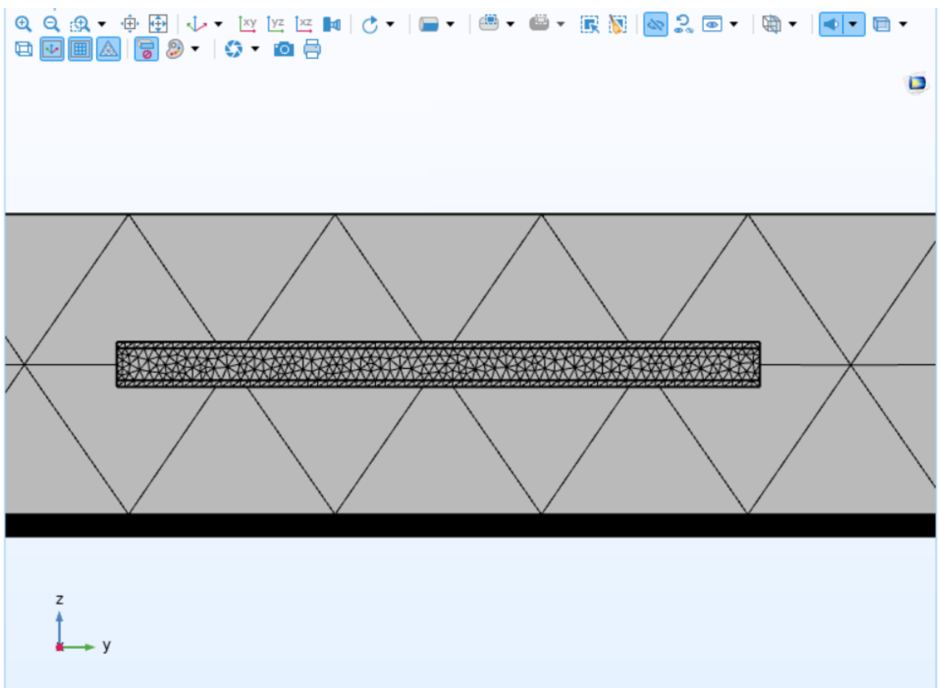


Fig. 3: Mesh cross section of the inner part.

Why is the mesh so different?

The mesh concentrates at the middle part, more nodes are created within this part to break up the model into smaller elements. More equations are generated for each mesh element and the results will be more accurate.

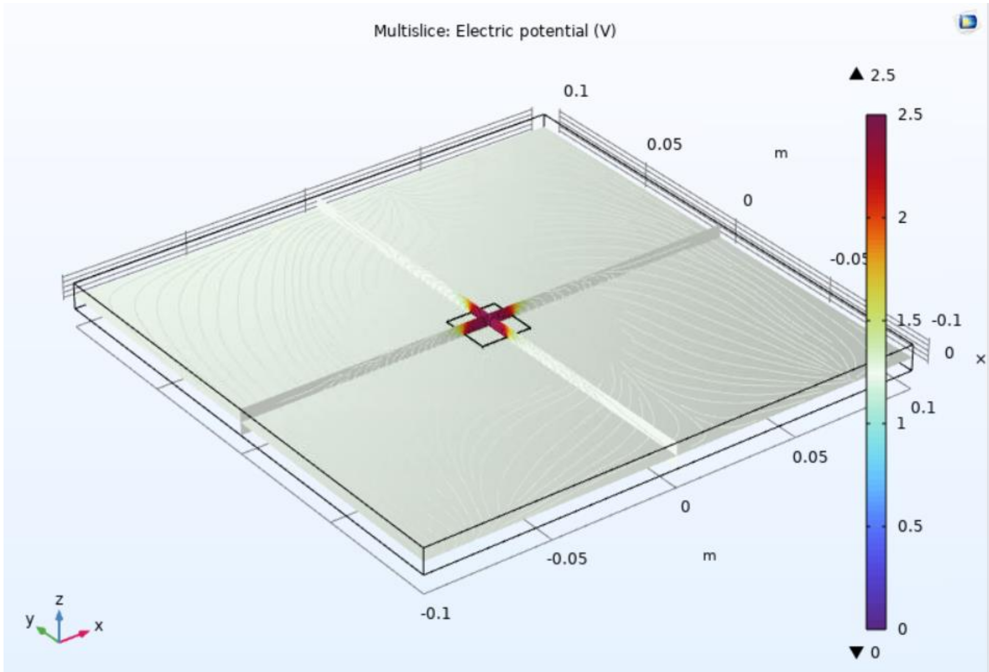


Fig. 4: Overall view of the potential distribution.

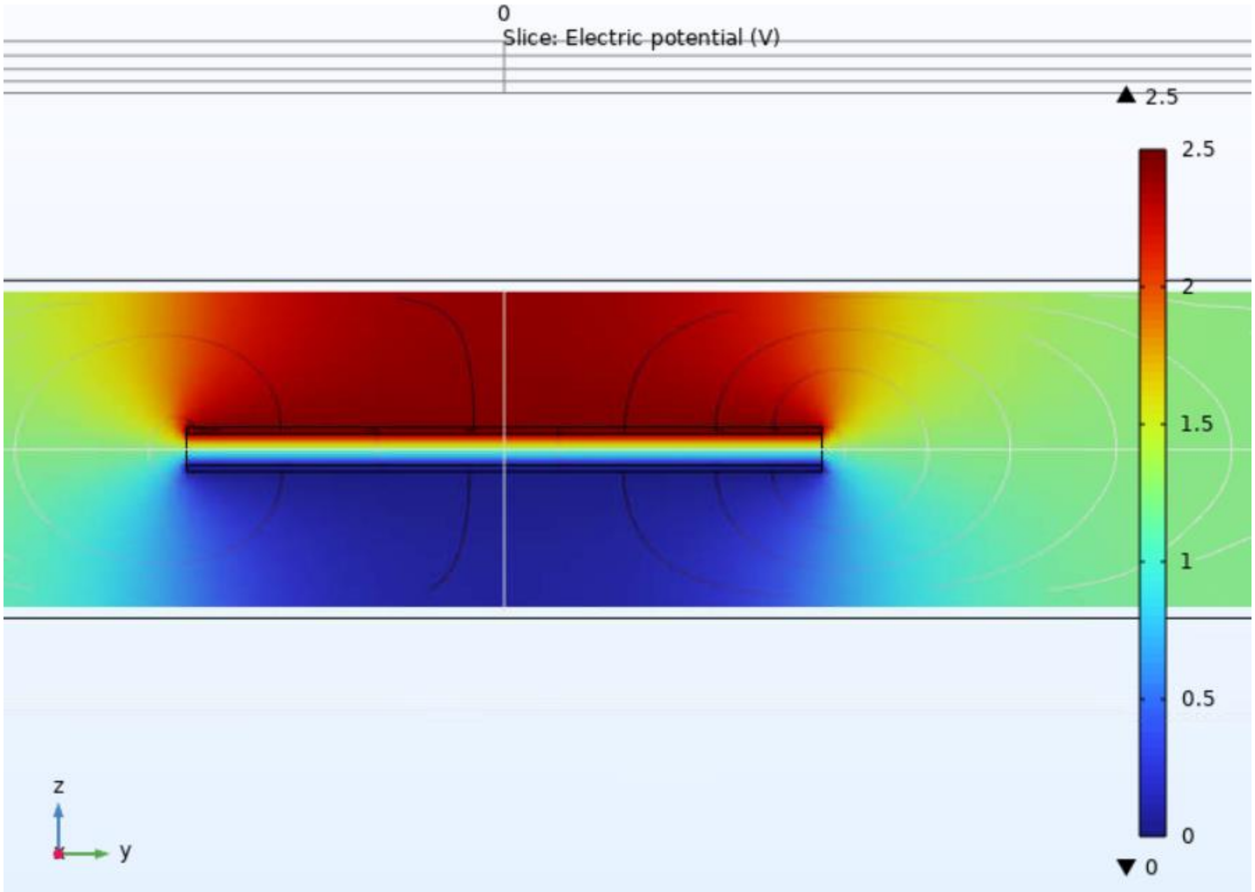


Fig. 5: The middle region

Show and briefly describe. The middle region is interesting, as are the ends. Why?

The electrostatic potential drops quickly and gradually from 2.5V to 0V. The direction of the electric field is directly downwards in the middle region from higher potential to lower potential. At the ends, the direction of the electric field can be represented as in logarithmically scaled arrows which circles around the end from 2.5V to 0V.

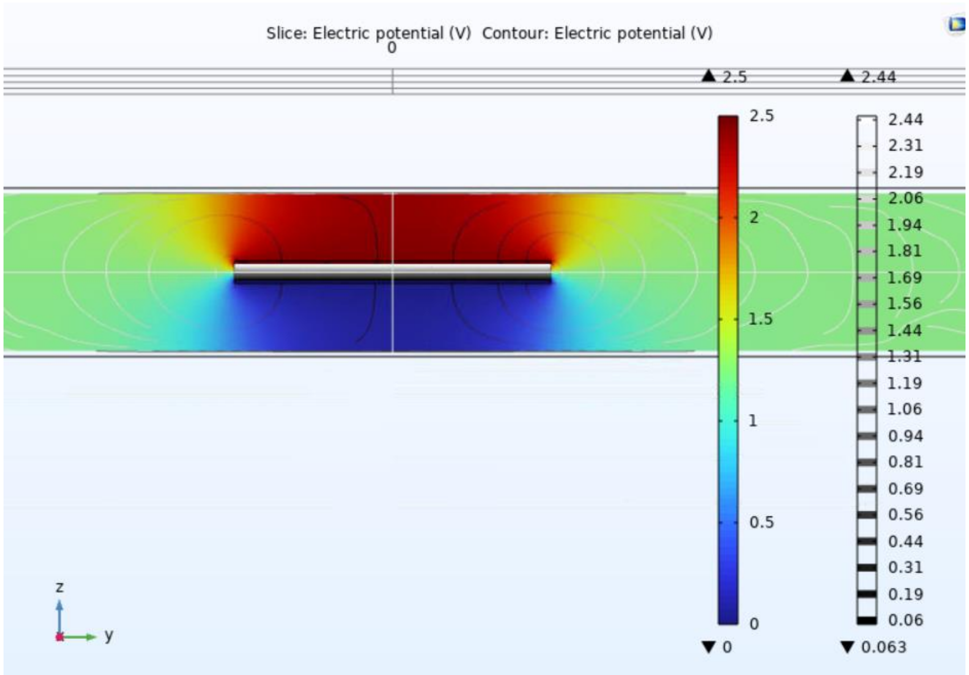


Fig. 6: Plot 5 with contour lines

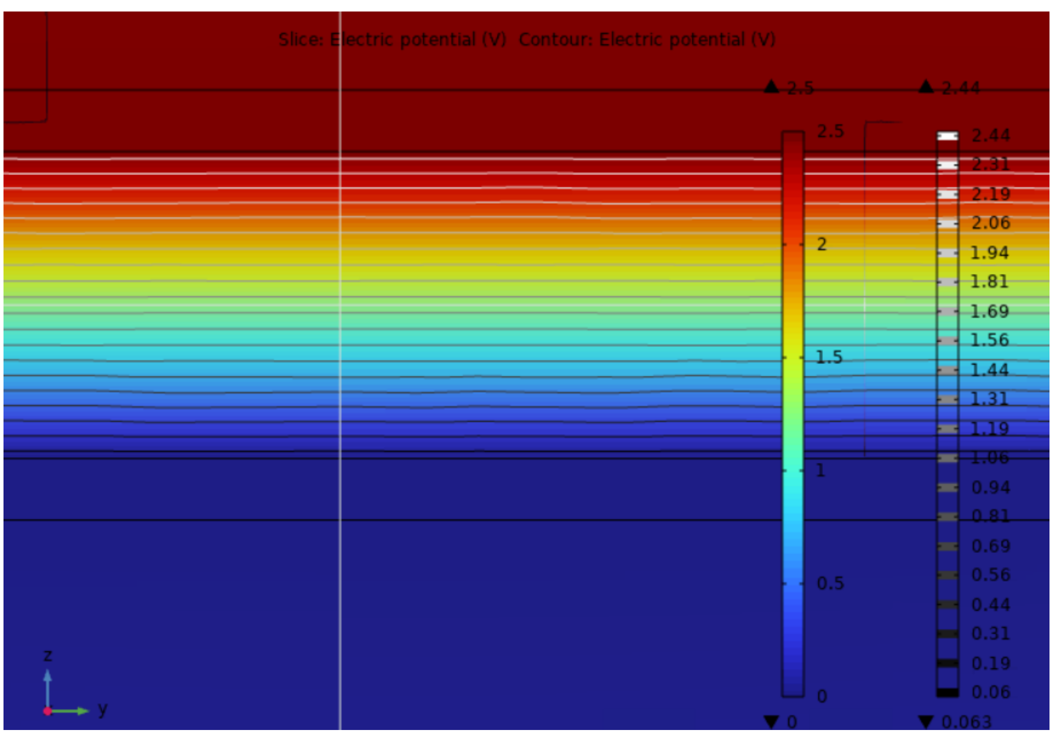


Fig. 7: Overall view of the potential distribution.

The drop of electric field between the parallel plates is linear, (the drop rate is constant), corresponding to the formula E= ΔV/Δr.

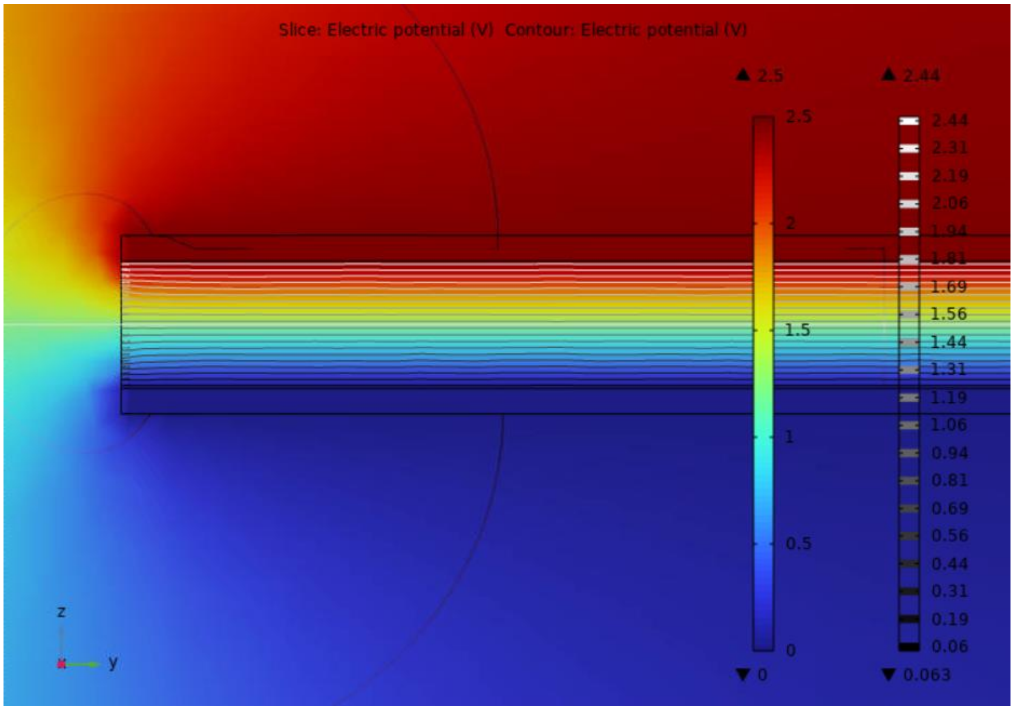


Fig. 8: Overall view of the potential distribution.

Calculate the total charge on the top plate:

Using COMSOL we get:

Surface charge density (C)
5.4379E-11

How does it compare with the analytic expression in Problem 1? 8.854×10^{-12}

It is larger than the solution in analytical derivation.

Which is more accurate?

The COMSOL result is more accurate, because analytic was looking at the infinite solution.

Now integrate the Q just on the bottom surface of the top plate.

Surface charge density (C)
5.3326E-11

How does it compare with the full Q?

It is the largest portion of it!

Now integrate Q on the full bottom plate.

Surface charge density (C)
-1.0501E-12

How does that compare with the Q on the top?

It is smaller than the Q on the top.

Part 3: 2D electrostatic simulation

How many elements: 20372

```
Number of vertex elements: 12
Number of boundary elements: 606
Number of elements: 20372
Minimum element quality: 0.5968
```

Much smaller than the 3D elements.

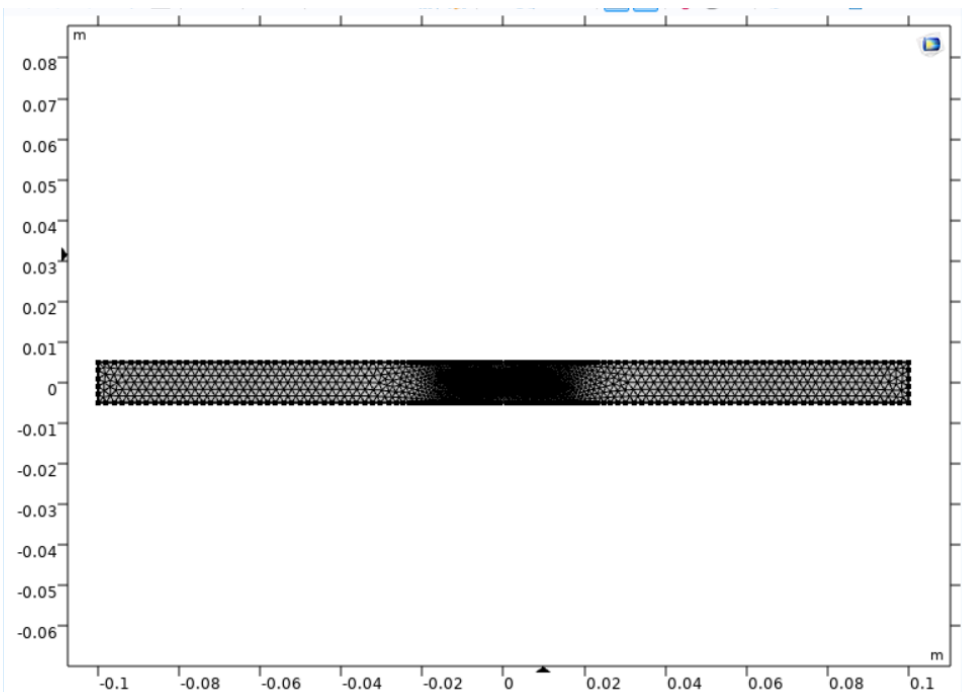


Fig. 9: Computational mesh for 2D simulation using extra fine

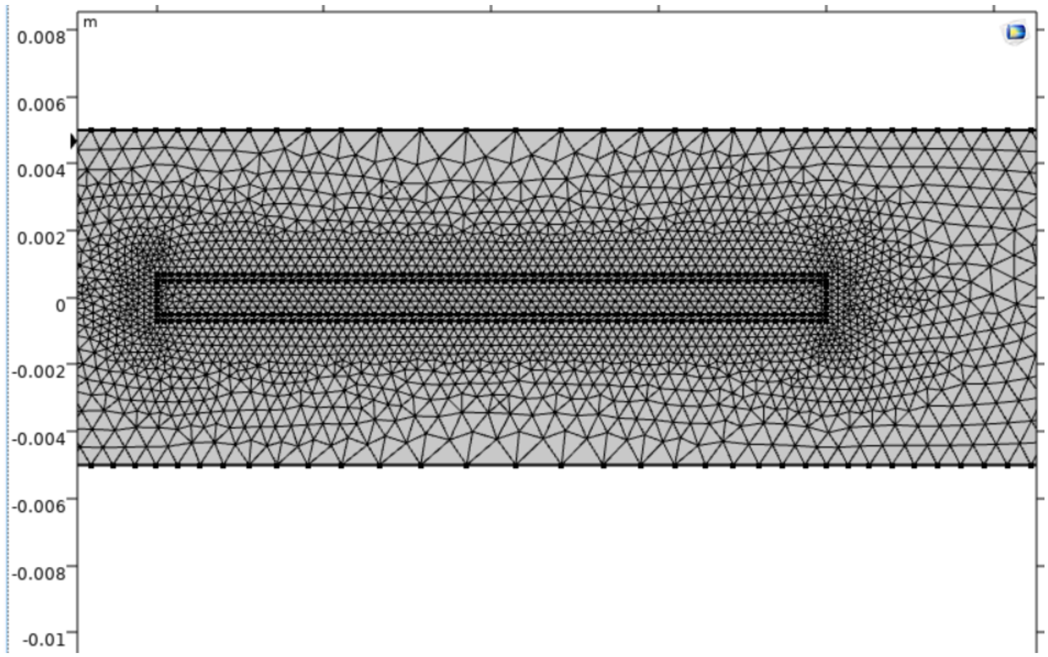


Fig. 10: Zoomed in view of Fig. 9

Similarly, the mesh concentrates at the middle part, more meshes are needed to solve analytical equations and provide a more accurate simulation result of electric potential.

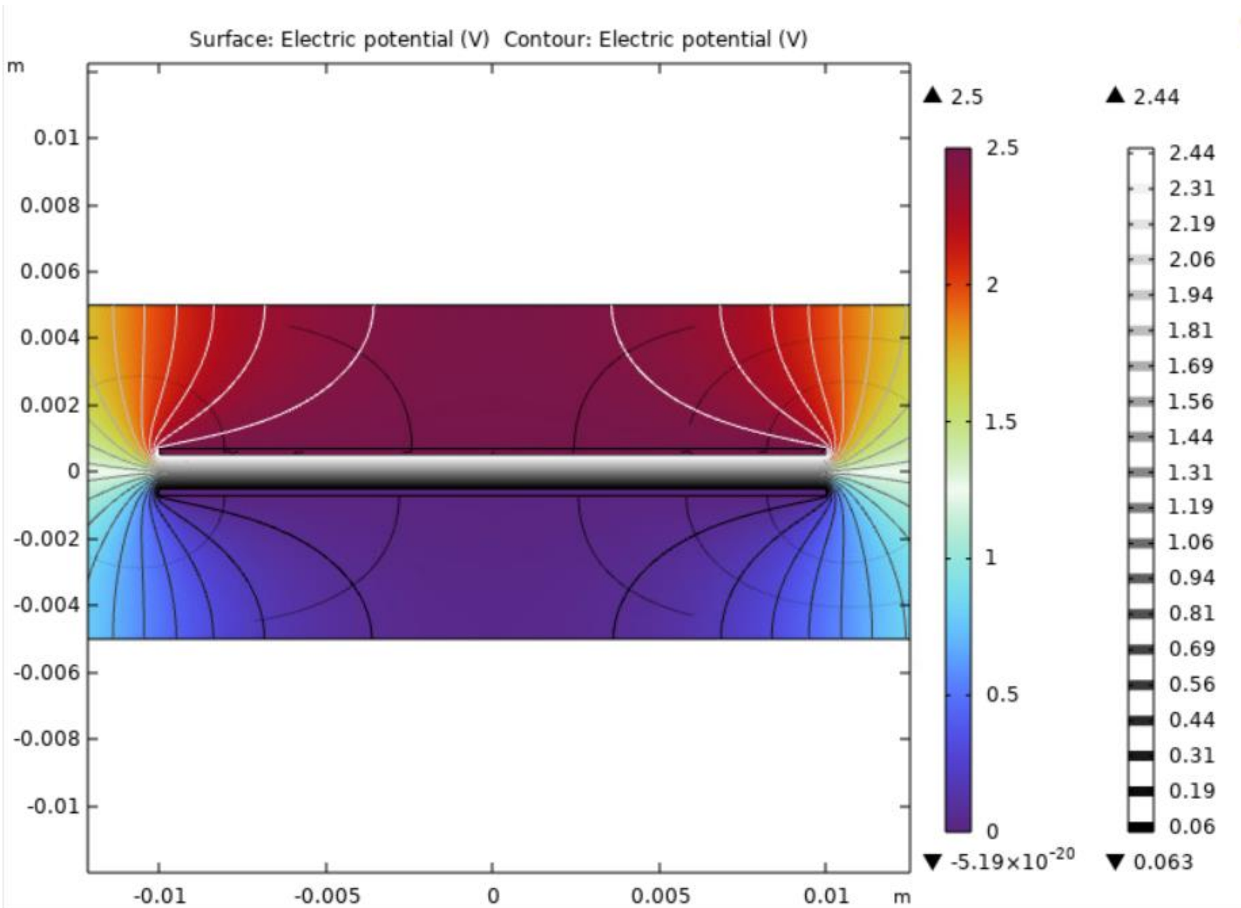


Fig. 11: Electric potential with contour lines

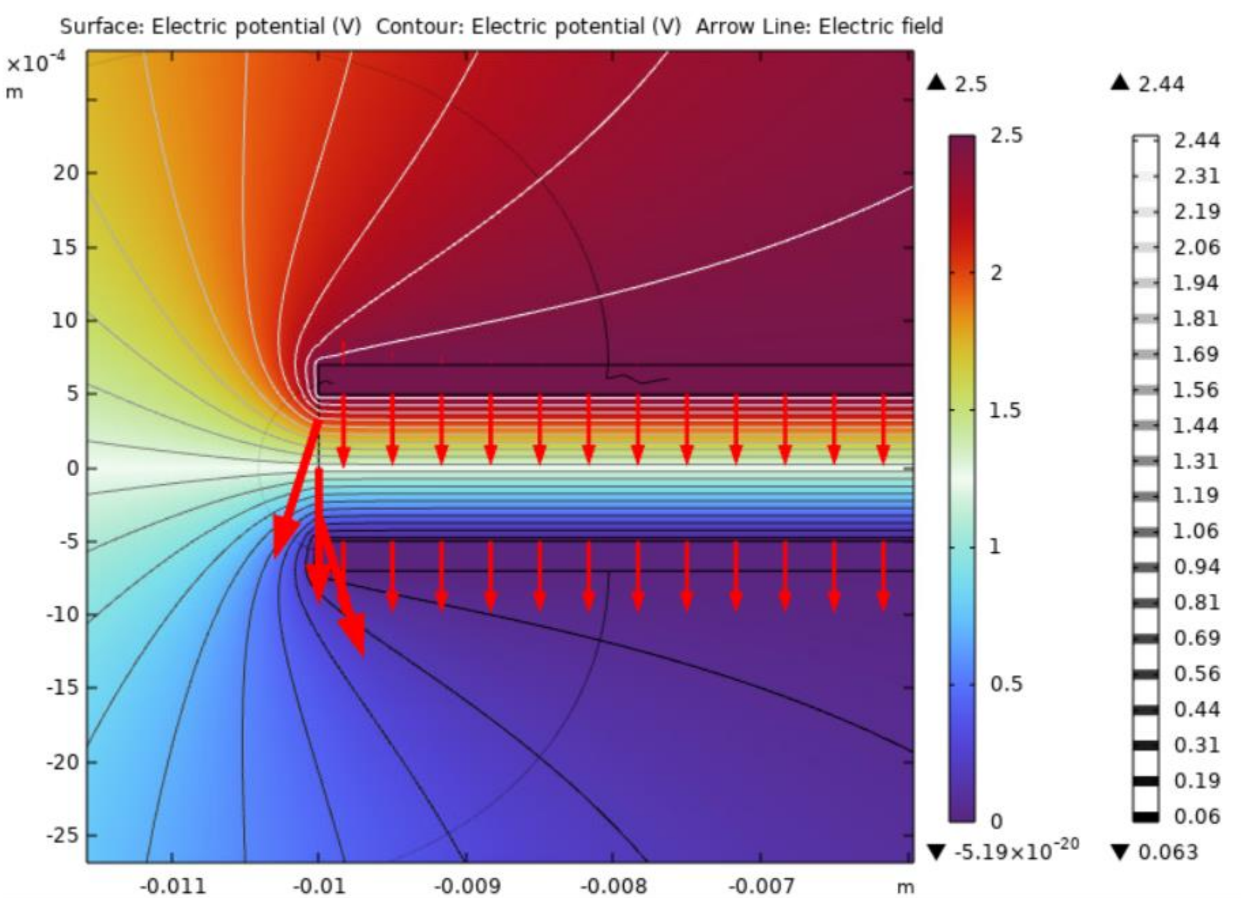
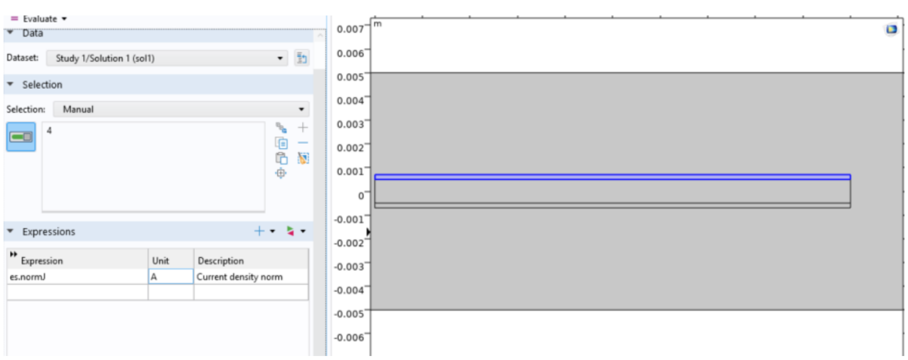


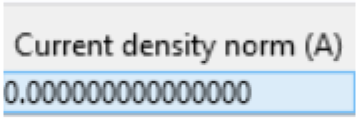
Fig. 12: Fig. 11: zoomed in with electric field lines.

In figures 8 and Fig. 12. We are plotting contour lines which stand for equal electric potential lines. In figure 12 we also plot the electric field using red arrows.

Integrating to obtain the charge over the top rectangle:



Getting zero:



Current density norm (A)
0.000000000000000

Does it equal the analytic value from Part 1, or the 3D value from Part 2?

No, the analytical value is different. Analytic solution being non linear, and the COMSOL distributing the temperature between mesh points linearly, so there is a subtle difference.

The 3D value should be more accurate than the 2D value and analytical solution value. More dimensions of electric potential simulation are demonstrated and more meshes are used.

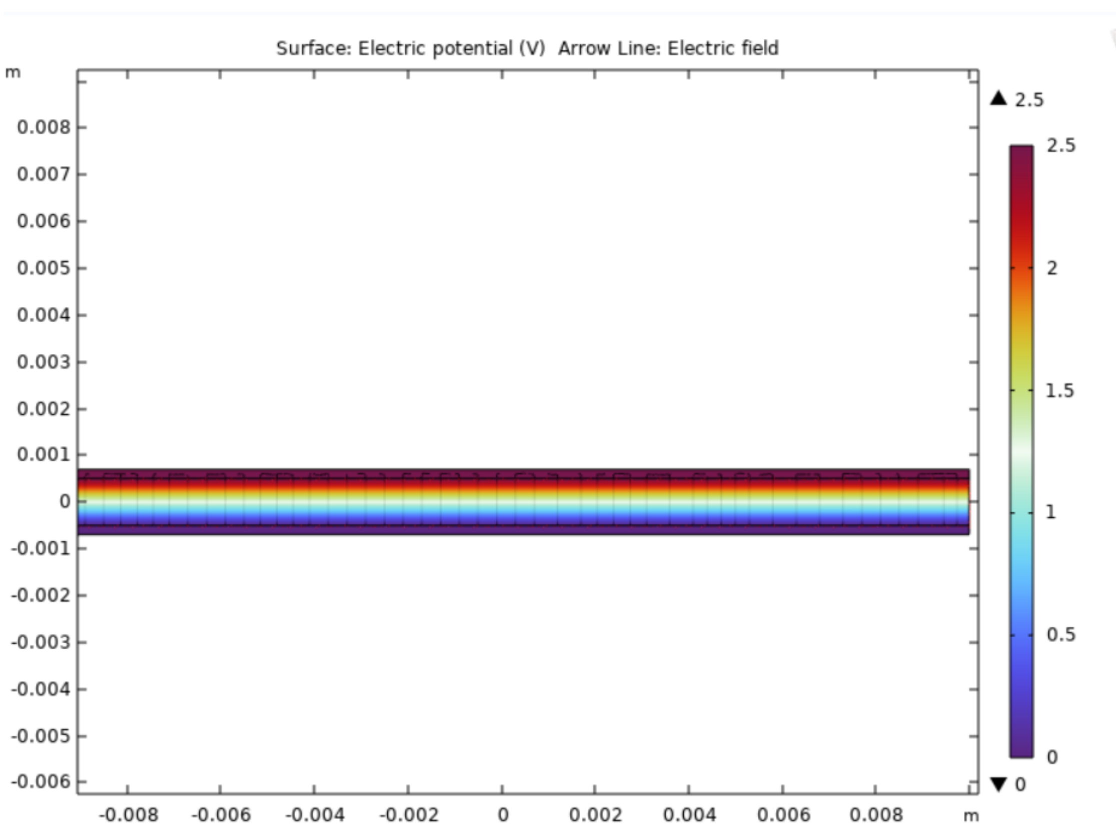


Fig. 13: Overall view of the potential distribution.

If we change the length from 0.02m to 0.01m, even to 0.00001m, other than getting shorter, the distribution of electrostatic potential will still look the same, there will be a constant potential region (0V and 2.5V) on each side and the drop of potential will be linear.

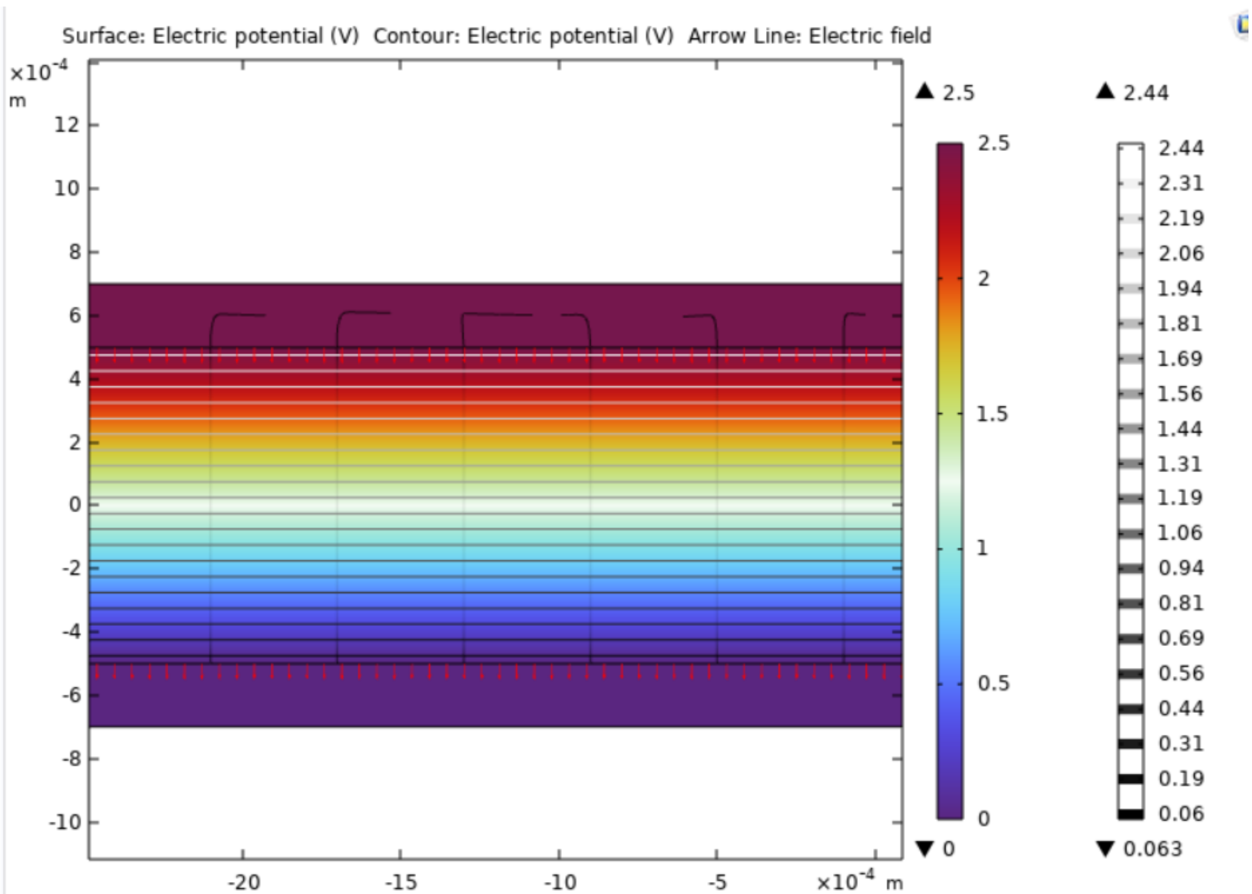


Fig. 14: Overall view of the potential distribution.

Current density norm (A)
0.000000000000000

It equals the first value of C in 2D. Of the 3 values of C, the one in 3D is closer to the analytical value as more dimensions of electric potential simulation are demonstrated and more meshes are used.

Conclusion:

In conclusion we have run a simulation of electrostatics. The value of capacitance C is calculated under different scenarios, e.g in analytical solution by derivation, in 3D model and 2D models. Compared to the results, the 3-D appears to be the most accurate run, however the 2-D and Analytical solution were also accurate but not as good as the 3-D model due to the lack in dimensional accuracy and less number of meshes used. We also found the distribution of electrostatic potential between and around the end of parallel plates shown by the contour lines.